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Solving Systems of Linear Algebraic Equations Using the Monte Carlo Method

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ABSTRACT

Mathematics is a vast science that has developed in different parts according to human needs and at different times to solve human problems since its inception. The branches of mathematics can be considered as an independent field due to their complexity. Numerical calculations or numerical analysis, in turn, can be found in different parts due to the calculations of equations that do not have real solutions. It can find their numerical solutions.

Numerical analysis can be used in various areas such as power calculations, logarithms, limit calculations, numerical derivatives of functions, numerical integrals, etc. mathematical calculations. In this regard, mathematicians have conducted research in various areas and after a period of time have reached a solution to the problem at hand and have left a mark in the form of various relations, formulas, and theorems, each of which has created convenience for the reader. One of these methods that can easily solve equations is solving a system of linear equations using the Monte Carlo method. Basically, this method, with its complexity, considering probabilistic calculations and obtaining the prices of a system and using mathematical hope, can easily solve a system of linear algebraic equations.

Keywords- Linear Algebraic Equations, Mathematical Hope, Linear equation system, Random variable, Eigenvalues and eigenvectors of each matrix.

I. INTRODUCTION

Mathematical Hope:

Definition 1: If a random variable x has a probability function f(x), then the mathematical expectation of x is

If X is discrete $$E(X)=\sum_x x f_X(x)$$

If X is continuous. $$=\int_{-\infty}^{\infty} x f_X(x)\,dx$$

Definition: If a random variable X has a probability distribution f(x) (probability density function f(x)), then the mathematical expectation of any function of X such as h(x) is

If X is discrete $$E\left(h(x)\right)=\sum_x h(x) f(x)$$

If X is continuous. $$=\int_{-\infty}^{\infty} h(x) f(x)\,dx$$

Example: A fair die is thrown once. If the random variable X represents the number that appears, determine the mathematical expectation of the random variable $[\![Y = 2X]\!]$ ^2 − 5.

Solution: The probability distribution f(x) is:

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

And according to the above definition

$$E(X)=\sum_{x=1}^{6}\left(2x^2-5\right)f(x)=\frac{1}{6}\left(-3+3+13+27+45+67\right)=\frac{76}{3}$$

Definition: If X and Y are two random variables with joint probability distribution $f_{X,Y}(x,y)$ (joint

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probability density function $f_{X,Y}(x,y)$). The mathematical expectation of any function of X and Y, such as $h(X,Y)$, is:

If X and Y are both discrete

$$E(h(X,Y)) = \sum_x \sum_y h(x,y) f_{X,Y}(x,y)$$

If X and Y are both continuous

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) f_{X,Y}(x,y)\, dxdy$$

Investigating and finding answers to any scientific phenomenon requires an appropriate algorithm for that phenomenon. For example, when we want to find the unknown function f given a number of points on the graph of that function, we search for a sequence of functions whose limit tends towards f.

$$\lim_{n\to\infty} f_n = f \qquad (1)$$

When the number of f_n is finite, this task is completed in a few steps; however, there are many phenomena and problems for which it is impossible to construct an algorithm to solve them. One way to escape this impasse is to use probability methods. Which is useful for investigating some mathematical and physical phenomena. In investigating these phenomena, the values of $f_n, \cdots, f_2, f_1$ obtained through a series of random experiments tend towards f. So, when $n \to \infty$ the random variable f_n which depends on the phenomenon converges to f; then for every $\varepsilon > 0$

$$\lim_{x\to\infty} P\left(\left|f - f_n\right| < \varepsilon\right) = 1 \qquad (2)$$

The value f is the probability of a random event, or the mathematical expectation that it will occur. f is called the random variable. f_n is found by n random experiments, which may involve assumptions. As is evident, in this approach, there is not much calculation involved and the requirements are achieved with the help of experimentation.

Methods that use random values are grouped under the general name of Monte Carlo method. In other words, Monte Carlo method refers to a set of methods with the help of which we can obtain some physical or mathematical phenomena by random experiments. The effectiveness of Monte Carlo method has become more widespread with the advent of computers.

Obtaining a proper and accurate estimate requires repeated and frequent calculations, but by using the Monte Carlo method, there is no longer any need to know how the desired and the found, or the given value and the searched value, are related.

Some of the mathematical phenomena that Monte Carlo simulation is used to solve are as follows.

Solving systems of linear equations, finding the inverse of a matrix, finding the eigenvalue and eigenvectors of each matrix, calculating multiple integrals, solving Dirichlet problems, etc. This method is also used to investigate many physical phenomena.

In this course, we will discuss calculating multiple integrals and finding the solution of linear equations using the Monte Carlo method. Consider the following linear system.

$$\sum_{j=1}^{n} a_{ij} x_j \quad (i = 1, 2, 3, \cdots, n) \qquad (1)$$

In some methods, we write the device (1) in the following special form:

$$X_i = \sum_{j=1}^{n} a_{ij} x_j + \beta_i \qquad (2)$$

By placing:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \quad \alpha = \left[\alpha_{ij}\right] \qquad (3)$$

We write the device (2) as follows:

$$X = \alpha X + \beta \qquad (4)$$

And we assume that all eigenvalues of matrix a are smaller than unity. In the special case, for canonical norms of matrix a, it is sufficient that the following condition.

$$\|\alpha\| < 1 \qquad (5)$$

is valid. So, the system (4) has a single solution. Which is obtained by the iteration method. Now a set of coefficients V_{ij} and also numbers P_{ij} is defined by the following equations

$$\alpha_{ij} = P_{ij} V_{ij} \qquad (ij = 1, 2, 3, \cdots, n) \qquad (6)$$

We choose which has the following condition.

$$P_{ij} \geq 0 \qquad P_{ij} > 0 \qquad (\alpha_{ij} \neq 0) \qquad (7)$$

$$\sum_{j=1}^{n} P_{ij} < 1 \qquad (i = 1, 2, 3, \cdots, n) \qquad (8)$$

Suppose

$$P_{i,n+1} = 1 - \sum_{j=1}^{n} Pij \qquad (i = 1, 2, 3, \cdots, n) \qquad (9)$$

We add:

Consider a particle with random behavior and states $S_1, S_2, \cdots, S_n, S_{n+1}$. (The probability of going from stage S_i to stage $P_{ij} = S_j$) In special states $S_{n+1} = T$ corresponds to the particle completely stopping. We also have:

$P_{n+1,j} = S_j$ In the state S_{n+1} Probability of transition from state $= 0\,(j = 1, 2, 3, \cdots, n)$

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$P_{n+1,j} = S_j$ In the state S_{n+1} Probability of transition from state $= 1 (j = n+1)$

Thus, this process of random behavior ends as soon as the particle reaches a finite state of T. This variation is commonly called a Markov chain with a finite number of states.

P_{ij} is called the transition probability? And matrix P is the matrix of steps $\{S_i\}$.

$$\pi = [■(p\_11\&p\_12\& \cdots \&p\_1n\&p\_1(n+1) @p\_21\&p\_22\& \cdots \&p\_2n\&p\_2(n+1) @ \vdots \& \vdots \& \cdots \& \vdots \& \vdots @p\_n1\&p\_22\& \cdots \&p\_nn\&p\_n(n+1) @0\&0\& \cdots \&0\&1)]" \quad "(10)$$

Let $S_i \left(i < n+1\right)$ be a fixed state different from the limit state. The random behavior of a particle that starts its motion at $S_i = S_{i_0}$, then moves to intermediate states $S_{i_1}, S_{i_2}, S_{i_3}, \cdots, S_{i_{n-1}}, S_{i_n}$ and ends its motion at limit $S_{i_{n+1}} = T$. We consider. The sum of states:

$$T\_i = \{S\_(i\_0), S\_(i\_1), S\_(i\_2), \cdots S\_(i\_n), S\_(i\_(n+1))\}" \quad "(11)$$

For brevity it will be called a path. Let X_i be a random value that depends on the random paths T_i starting from state S_i. And let 2 be the value that takes.

$$\xi(T\_i) = \beta\_(i\_0) + V\_(i\_0\, i\_1)\, \beta\_(i\_1) + V\_(i\_1\, i\_2)\, \beta\_(i\_2) + \cdots + b\_(i\_o\, i\_1) + V\_(i\_(m-1)\, i\_m)\, \beta\_(i\_m)" \quad "(12)" \quad "$$

where $\beta_i = \left(i = i_0, i_1, i_2, \cdots, i_m\right)$ are the corresponding constant terms of system (2)?

In the special case if is $V_{ij} = 1$. We simply have that:

$$\xi(T\_i) = \beta\_(i\_0) + \beta\_(i\_1) + \beta\_(i\_2) + \cdots + \beta\_(i\_m)" \quad "(13)$$

By the law of product of direct probabilities, the probability of random path T_i occurring and thus producing value $x\left(T_i\right)$ is:

$$P(T\_i) = P\_(i\_0\, i\_1) + P\_(i\_1\, i\_2) + P\_(i\_2\, i\_3) + \cdots + P\_(i\_m\, i\_(m+1))" \quad "i\_(m+1) = n+1, " \; " i\_0 = i" \quad "(14)$$

Theorem: The mathematical expectations of $Mx_i = x_i \left(I = 1, 2, \cdots, n\right)$ are the roots of system (2).

Proof: Paths T_i starting from state S_i may be divided into $n+1$ categories that depend on the first stage.

$$■(T\_(i\_1) = \{S\_i, S\_1, S\_(i\_1) \cdots\} @T\_(i\_1) = \{S\_i, S\_2, S\_(i\_1) \cdots\} @ \quad \ldots" \quad " \ldots " \quad " \ldots " \quad " \ldots " (15) @T\_(i\_n) = \{S\_i, S\_n, S\_(i\_1) \cdots\} @T\_(i\_(n+1)) = \{S\_i, S\_(n+1)\})$$

A particle that starts a random behavior at stage S_i can go to stage S_2, etc., and once a certain number of stages have been completed, its random behavior ends at the boundary.

If a particle has path 1 and 2, then

$$■(\xi(T\_(i\_j)) = \beta\_i + V\_ij\, \beta\_j + V\_ij\, V\_ij\, \beta\_(i\_2) + \cdots + V\_ij\, V\_(ji\_2) \cdots V\_(i\_(m-1) - i\_m)\, \beta\_(i\_m) @" \quad " = \beta\_i + V\_ij\, (\beta\_j + V\_(ji\_2)\, \beta\_(i\_2) + \cdots + V\_(ji\_2)) + V\_(i\_(m-1) - i\_m)\, \beta\_(i\_m) = \beta\_i\, \xi(T\_i)" \quad "(16))$$

Where t_i is a path starting from stage S_j.

When a particle reaches region T and its path is in arrangement $\left\{S_{i,n+1}\right\} = T_{i,n+1}$, then the probability that path T_i is a path of type T_{ij} is clearly equal to P_{ij}. By the definition of mathematical expectation, we have:

$$MX\_i = \sum\_(T\_i) ▒ \xi(T\_i) P(T\_i) = \sum\_j " \; " \sum\_(T\_ij) ▒ \xi(T\_ij) P(T\_ij)" \quad "(17)$$

If $j < n+1$ is not, then T_{ij} includes interval $\left(S_i, S_i\right)$ and path T_i, therefore;

$$P(T\_ij) = P\_ij\, p(T\_j)" \quad "(19)$$

So, according to equation (16), it is obtained as follows.

$$MX_i = \sum_{j=1}^{n} \sum_{\eta} \left[\beta_i + V_{ij}\xi\left(T_i\right)\right] \cdot P_{ij} P\left(T_i\right) + \beta_i P_{i,n+1} \qquad (20)$$

Or will we have

$$MX_i = \sum_{i=1}^{n} P_{ij} V_{ij} \sum_{T_j} \xi\left(T_i\right) P\left(T_j\right) + \beta_i \left[\sum_{i=1}^{n} P_{ij} \sum_{T_j} \left(T_j\right) + P_{i,n+1}\right] \qquad (21)$$

In addition, we have

$$\sum_{T_j} P\left(T_j\right) = 1 \qquad (22)$$

$$\sum_{j=1}^{n} P_{ij} \sum_{T_j} P\left(T_j\right) + P_{i,n+1} = \sum_{j=1}^{n+1} P_{ij} \qquad (23)$$

Therefore

$$MX_i = \sum_{j=1}^{n} \alpha_{ij} MX_i + \beta_i \qquad \left(i = 1, 2, 3, \cdots, n\right) \qquad (24)$$

That

$$\alpha_{ij} = P_{ij} V_{ij} \qquad (25)$$

And thus, the proof of the theorem is complete.

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In proving this theorem, we assumed that the mathematical expectation is $x_i = MX_i$. Because when condition (5) holds, the random values have a finite mathematical expectation of x_i. And this can be proved.

To experimentally determine the value of $N, x_i = MX_i$, consider the random behavior $T_i^{(k)}$ with random paths where $(k = 1, 2, 3, \cdots, N)$ and the initial state (S_i) are considered, and record the values $\xi\left(T_i^{(k)}\right)$ of the random value x_i each time.

Suppose that the trials are independent, that the value of $N, x_i = MX_i$ has a finite variance. By Sheff's left theorem, and for N sufficiently large, the following inequality will be true with arbitrary probability close to I.

$$\left| x_i - \frac{1}{N}\sum_{k=1}^{N} \xi\left(T_i^{(k)}\right) \right| < \varepsilon \qquad (26)$$

(Which is a limiting error.)
Therefore, the roots of system (2) can be approximately determined by the following formula.

$$x_i = \frac{1}{N}\sum_{k=1}^{n} \xi\left(T_i^{(k)}\right) \qquad (27)$$

In a special case, this method can be used to find a matrix like the one below.

$$A = E - a \qquad (28)$$

where $\|\alpha\| < 1$ and $E = \left[d_{ij}\right]$ are a unit matrix.

Note that the elements of the matrix $A^{-1} = \left[x_{ij}\right]$ are the roots of the system

$$\sum_{k=1}^{n} \left(\delta_{ik} - \alpha_{ik}\right) x_{kj} = \delta_{ij} \qquad (i, j = 1, 2, 3, \cdots n)$$

where the elements of each column $x_{1j}, x_{2j}, \cdots x_{nj}$ $\quad j = 1, 2, 3, \cdots, n$ of matrix A^{-1} are represented by the linear system:

$$x_{ij} = \sum_{k=1}^{n} \alpha_{ik} x_{kj} + \delta_{ij} \qquad (i = 1, 2, 3, \cdots, n)$$

They have been determined.

Based on what was said, we start from state $S_i = S_{i0}$ and obtain the random value j with the following values for fixed X_i.

$$\xi_j\left(T_i\right) = \delta_{i_o j} + \delta_{i,j} V_{i_0 i_1} + \cdots + \delta_{i_n i} V_{i_0 i_1} \cdots V_{i_{n-1}} V_{i_n} \qquad (30)$$

where $T_i = \left\{S_{i_0}, S_{i_1}, S_{i_2}, \cdots, S_{i_m}, S_{i_{m+1}} = T\right\}$ and the number V_{ij} are like P_{ij}. and are obtained from equations $\alpha_{ij} = P_{ij} V_{ij}$., are such that P_{ij} shows the probability of transition from state S_i to S_j. The mathematical expectations MX_i give us the elements of matrix A^{-1}.

Now we will practically show how to organize a random behavior. For this purpose, we consider the random behavior of a particle with transition probabilities P_{ij}. For simplicity, we assume that they P_{ij} are decimal functions with common denominator 10^s (s natural numbers).

$$P_{i_1} = \frac{t_{i1}}{10^s}, \quad P_{i_2} = \frac{t_{i2}}{10^s}, \cdots, P_{i,n+1} = \frac{t_{i,n+1}}{10^s} \qquad (31)$$

where 1 are non-negative integers

$$t_{i_1} + t_{i_2} + t_{i_2} + \cdots + t_{i_n} + t_{i_{n+1}} = 10^s \qquad (i = 1, 2, 3, \cdots, n) \qquad (32)$$

Consider a particle that starts moving from initial state S_i, let $\{x\}$ be a set of s-digit numbers less than unity with a uniform distribution over the interval $[0,1]$. For example, consider the black elements of random numbers. Suppose we generate a random number x. If the inequality $0 \le x < \frac{t_{i_1}}{10^s}$ holds, then we obtain that the particle goes from S_i to S_i. Other transitions are also obtained in a similar way. In special cases, if the random number x is such that

$$\frac{t_{i_1} + t_{i_2} \cdots + t_{i_n}}{10^s} \le x \le \frac{t_{i_1} + t_{i_2} \cdots + t_{i_n} + t_{i_{n+1}}}{10^s} \qquad (32)$$

(28)

The particle hits area $S_{n+1} = T$.

Based on the obtained agreement, it is clear that the number of sentences of the desired states for transition $S_i \to S_j$ is proportional to $(i = 1, 2, \cdots, n+1)$. With the corresponding numbers $t_{i_1} + t_{i_2} \cdots + t_{i_n} + t_{i_{n+1}}$, these states have equal probabilities. Therefore, the transition probability is as follows:

(29)

$$P\left(S_i \to S_j\right) \quad \frac{t_{i_1}}{10^s} = P_{ij} \qquad (i = 1, 2, 3, \cdots, n,\ j = 1, 2, 3, \cdots, n+1) \qquad (33)$$

By extracting a sequence of random numbers and obtaining the above rule, we obtain the random behavior of a particle with a fixed initial state and find the transition matrices. To obtain the correctness of the roots, we must consider a sufficiently large number k for independent random behaviors.

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Example: Solve the following system of equations using the Monte Carlo method.

$$\begin{cases} x_1 = 0.1x_1 + 0.2x_2 + 0.7 \\ x_2 = 0.2x_1 + 0.3x_2 + 1.1 \end{cases}$$

We assume that

$$V_{21} = 1 \qquad V_{22} = -1 \qquad V_{11} = 1 \qquad V_{12} = 1$$

Therefore, the transition probability matrix is as follows.

$$\pi = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.1 & 0.3 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

The elements of the first row are the transition probabilities from state S_i to S_2, S_1 and $S_3 = T$, respectively, and the elements of the second row are the transition probabilities from state S_2 to S_3, S_2, S_1, and the margin corresponds to the range T. Because the elements of the matrix P are multiples of 0.1. And one of the cases that can be considered are single-digit random numbers whose digits are obtained from some random sequences and are given in $\|\alpha\| < 1$.

The results collected in the relation are for 20 random behaviors with initial state S_1. Random number x satisfies the transitions of the steps using the following guidelines.

For the initial stages of S_1:

1. If $0 \le x \le 0.1$ Then $S_1 \to S_1$
2. if $0.1 \le 0.3$ then $S_1 \to S_2$
3. if $0.3 \le x < 1$ then $S_1 \to T$

For the initial stages of S_2:

1. if $0 \le x \le 0.2$ then $S_2 \to S_1$
2. if $0.2 \to X \to 0.5$ then $S_2 \to S_2$
3. if $0.5 \to X \to 1$ then $S_2 \to T$

The values of X_1 calculated by the command (10) are updated in the last column of equation $\|\alpha\| < 1$. As a result, we have:

$$x_1 = MX_1 \approx \frac{1}{20}\left(21 \times 0.7 + 4 \times (1{,}1)\right) = 0.96$$

The unknown x_2 is also obtained in a similar way. Note that the roots of the system

$$\begin{cases} x_1 = 0.1x_1 + 0.2x_2 + 0.7 \\ x_2 = 0.2x_1 + 0.3x_2 + 1.1 \end{cases}$$

is equal to $x_2 = 1, x_1 = 1$.

Other methods can also be used to solve linear algebraic equations using the Monte Carlo method. As we mentioned earlier, the fourth column of the determinant, i.e. the values of x_i, is calculated using the formula (34). For example, we have calculated an example of it below, and the remaining values can be easily investigated in a similar way.

$$\xi(T_i) = \beta_{i_0} + V_{i_0 j_1}\beta_{i_1} + V_{i_1 j_2}\beta_{i_2} + \cdots + V_{i_0 j_2} \cdots V_{i_{m-1} j_m}\beta_{i_m}$$

Now consider list number 12, in which we have:

$$S_1 \to S_1 \to S_2 \to S_2 \to S_1 \to S_1 \to T$$

Which we will have:

$$i_0 = 1, i_1 = 1, i_2 = 2, i_3 = 2, i_4 = 1, i_5 = 2, m = 5$$

So:

$$x(T) = \beta_1 + V_{11}\beta_1 + V_{11}V_{12}\beta_2 + V_{11}V_{12}V_{22}\beta_2 + V_{11}V_{12}V_{22}V_{21}\beta_1 + V_{11}V_{12}V_{22}V_{21}V_{12}\beta_2 \qquad (35)$$

That by placing

$$\xi(T) = 0.7 + (1)(0.7) + (1)(1)(1.1) + (1)(1)(-1)(1.1) + (1)(1)(-1)(1)(0.7) + (1)(1)(-1)(1)(1)(1.1)$$
$$= 0.7 + 0.7 + 1.1 - 0.7 - 1.1$$

You should note that the transitions of $S_i \to T$ easily correspond to the value of $\xi(T_i) = \beta_i$, and we saw this in the proof of the theorem.

In Table 1-4, we have extracted the random values from Table 1-1 for the random numbers. Recall again that the values of random variable 1 are the same as 2 and are listed in the fourth column.

Number	x random number	Random behavior path	X_i Random value
1	0.5	$S_1 \to T$	0.7
2	0.7	$S_1 \to T$	0.7
3	0.7	$S_1 \to T$	0.7
4	0.0	$S_1 \to S_1 \to T$	0.7+0,7
	0.5		
6	0.7	$S_1 \to T$	0.7
7	0.1	$S_1 \to S_1 \to T$	0.7+1.1
	0.8		
8	0.7	$S_1 \to T$	0.7
9	0.3	$S_1 \to T$	0.7
10	0.7	$S_1 \to T$	0.7
11	0.1		
	0.0	$S_1 \to S_2 \to S_1 \to T$	
12	0.0		
	0.3	$S_1 \to S_1 \to S_1 \to S_2$	0.7+0.7+1.1
	0.1	$S_1 \to S_2 \to T$	-1.1-1.7-1.1
	0.6		
13	0.9	$S_1 T$	0.7

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14	0.6	S_1T	0.7
15	0.1	$S_1 \to S_1 \to T$	0.7+1.1
	0.5		
16	0.3	$S_1 \to T$	0.7
17	0.3	$S_1 \to T$	0.7
18	0.2	$S_1 \to T$	0.7
	0.4		
	0.4	$S_1 \to S_2 \to S_2 \to S_2$	0.7+1,1-1.1
	0.1		+1.1-1.1-0.7
	0.6		
19	0.6	$S_1 \to T$	0.7
20	0.2	$S_1 \to S_2 \to T$	0.7+1.1
		$\sum$	$21\times0.7+4\times1.1$

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