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Non-Linear λ-Jordan Triple Derivation on Prime Algebras

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ABSTRACT

Let A be a prime $*-$algebra and Φ a $\lambda-$Jordan triple deriva- tion on A, that is, for every $A, B, C \in$ A, $\Phi(A \diamond B) = \Phi(A) \diamond B + A \diamond \Phi(B)$ wher $A \diamond_\lambda B = AB^* + \lambda BA$ such that a real scalar $|\lambda| \neq 0, 1$, and Φ is addi- tive. moreover, if $\Phi(I)$ and $\Phi(il)$ are selfadjiont then Φ is a $*-$derivation

***Keywords*-** Prime Algebras, λ-Jordan Triple Derivation, Non-Linearity.

I. INTRODUCTION

Let be a ring. For A, B, We denote by $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, the Jordan product and the Lie product, respectively. These products have recently attracted many authors' attention (for example, see[2, 7, 10, 11]). in addition, some authors have considered triple products of three elements. For example, the outhors in [4] considered two von Neumann algebras and such that one of them has no central abelian projections. Let $\lambda = 1$ be a non-zero complex number, and let Φ : be a, not necessarily linear, bijection with $\Phi(I) = I$. then, Φ preserves the following condation

$\Phi(A \diamond_\lambda B \diamond_\lambda C) = \Phi(A) \diamond_\lambda \Phi(B) \diamond_\lambda \Phi(C)$, (1.1)

for $A, B, C \in$ A if and only if one of the following statements holds:

- $\lambda \in$ R, and there exists a central projection $P \in$ A such that $\Phi(P)$ is a central projection in B, $\Phi|_{AP}$: A$P \to$ B$\Phi(P)$ is a linear $*-$isomorphism and $\Phi|_{A(I-P)}$: A$(I-P) \to$ B$(I-\Phi(P))$ is conjugate linear $*-$isomorphism.
- $\lambda \notin$ R, and Φ is a linear $*-$isomorphism.

the map Φ which holds in 1.1 preserves the λ Jordan triple product. we should note that λ is not necessarily associative. in order to clarify this, we mention that

$A \diamond_\lambda B \diamond_\lambda C := (A \diamond_\lambda B) \diamond_\lambda C = ABC + \lambda(BA^*C + CB^*A^*) + |\lambda|^2 CAB^*$ (1.2)

For more papers regarding maps preserving the triple product, the interested reader may refer [3, 5, 8, 12]

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we define λ Jordan product by $A \diamond_\lambda B = AB + \lambda BA^*$. we say that the map Φ (not necessarily linear) with the property of $\Phi(A \diamond_\lambda B) = \Phi(A) \diamond_\lambda B + A \diamond_\lambda \Phi(B)$ is a λ Jordan derivation map. it is clear that, for $\lambda = -1$ and $\lambda = 1$, the λ Jordan derivation map is a Lie derivation and a Jordan derivation, respectively [1]. we should mention here that, whenever we say Φ is a derivation, it means that $\Phi(AB) = \Phi(A)B + A\Phi(B)$.

Recently, Yu and Zhang in [14] proved that every non-linear Lie derivation from a factor von Neumann

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algebra into itself is an additive derivation. Also,Li, Lu and Fung [6] investigated a non-linear λ Jordan derivation. they showed that, if ( ) is a von Neumann algebra without central ablian projections and λ is a non-zero scaler, then
$\Phi : \mathcal{A} \to B(H)$
is a non-linear λ Jordan derivation if and only if Φ is an additive derivation. in [13], the authors showed that the Jordan derivation map, i.e., $\Phi(A \diamond_1 B) = \Phi(A) \diamond_1 B + A \diamond_1 \Phi(B)$, on every factor von Neumann algebra ( ) is an additive derivation.
the authors in [9] introduced the concept of Skew Lie triple derivations. Amap
$\Phi : \mathcal{A} \to \mathcal{A}$
is a non-linear Skew Lie triple derivation if
$\Phi([[A, B]_*, C]_*) = [[\Phi(A), B]_*, C]_* + [[A, \Phi(B)]_*, C] + [[A, B]_*, \Phi(C)]_*$
for all $A, B, C \in \mathcal{A}$, where $[A, B] = AB - BA^*$. They showed that, it Φ preserves that above charactrizations on factor von Neumann algebras, then Φ is additive $*-$derivation. in this paper, motivated by the above results, weconsider a map Φ on a prime $*-$algebra $\mathcal{A}$ which holds under the following conditions $\Phi(A \diamond B) = \Phi(A) \diamond B + A \diamond \Phi(B)$ where $A \diamond B = AB^* + \lambda BA$ is such that a real scalar $|\lambda| \neq 0, 1$, and Φ is additive. If $\Phi(I)$ and $\Phi(iI)$ are selfadjiont then Φ is a $*-$derivation. we say that $\mathcal{A}$ is prime, that is, for $A, B \in \mathcal{A}$, if $A\mathcal{A}B = \{0\}$, then $A = 0$ or $B = 0$.

1. Main Results

Our main theorem is as follows:

Theorem 2.1. *Let $\mathcal{A}$ be a prime $*$-algebra with unit I and a nontrivial projec-tion. Then, the map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies the following condition*
$\Phi(A \diamond B) = \Phi(A) \diamond B + A \diamond \Phi(B)$ (2.1)
where $A \diamond_\lambda B = AB^* + \lambda BA$ is such that a real scalar $|\lambda| \neq 0, 1$, is additive.

Proof. Let P_1 be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$, $i, j = 1, 2$, Then,

$$\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}.$$

For every $A \in \mathcal{A}$, we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all which follows, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. in order to show additivity of Φ on $\mathcal{A}$, we use the above partition of $\mathcal{A}$ and provide some claims wich prove that Φ is additive on each $\mathcal{A}_{ij}$, $i, j = 1, 2$.
the above theorem is proven by several claims.

Claim 1. *We show that* $\Phi(0) = 0$.
Proof. if $\Phi(0) = 0$, then, by successively putting $A = 0, B = 0$, and then,$C = 0$ in 1.2, we obtain a contradiction. □

Claim 2. *For each $A_{12} \in \mathcal{A}_{12}$ and $A_{21} \in \mathcal{A}_{21}$ we have*
$\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21})$.
Proof. We show that
$T = \Phi(A_{12} + A_{21}) - \Phi(A_{12}) - \Phi(A_{21}) = 0$.
we can write
$\Phi(A_{12} + A_{21}) \diamond (p_1 - p_2) + (A_{12} + A_{21})\Phi(p_1 - p_2)$
$= \Phi((A_{12} + A_{21}) \diamond (p_1 - p_2))$
$= \Phi(A_{12} \diamond (p_1 - p_2)) + \Phi(A_{21} \diamond (p_1 - p_2))$
$= (\Phi(A_{12}) + \Phi(A_{21})) \diamond (p_1 - p_2) + (A_{12} + A_{21}) \diamond \Phi(p_1 - p_2)$
thus we have
$T \diamond (p_1 - p_2) = 0$
Since $T = T_{11} + T_{12} + T_{21} + T_{22}$, then
$(1 + \lambda)T_{11} + (1 - \lambda)T_{21} - (1 - \lambda)T_{12} - (1 + \lambda)T_{22} = 0$
we know that $|\lambda| \neq 0, 1$ then
$T_{11} = T_{12} = T_{21} = T_{22} = 0$
□

Claim 3. *For each $A_{11} \in \mathcal{A}_{11}$, $A_{12} \in \mathcal{A}_{12}$, $A_{21} \in \mathcal{A}_{21}$ we have*
$\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})$.

Proof. we show that for T in $\mathcal{A}$ the following holds
$T = \Phi(A_{11} + A_{12} + A_{21}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) = 0$. (2.2)
we can write
$\Phi(A_{11} + A_{12} + A_{21}) \diamond (P_1 - P_2) + (A_{11} + A_{12} + A_{21}) \diamond \Phi(P_1 - P_2)$
$= \Phi((A_{11} + A_{12} + A_{21}) \diamond (P_1 - P_2))$
$= \Phi(A_{11} \diamond (P_1 - P_2)) + \Phi(A_{12} \diamond (P_1 - P_2)) + \Phi(A_{21} \diamond (P_1 - P_2))$
$= (A_{11} + A_{12} + A_{21}) \diamond \Phi(P_1 - P_2) \diamond (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})) \diamond (P_1 - P_2)$.
Then we have
since $T = T_{11} + T_{12} + T_{21} + T_{22}$ we obtian
$(1 + \lambda)T_{11} - (1 - \lambda)T_{12} + (1 - \lambda)T_{21} - (1 + \lambda)T_{22} = 0$
since $|\lambda| \neq 0, 1$ we have $T_{11} = T_{12} = T_{21} = T_{22} = 0$ □

Claim 4. *For each $A_{11} \in \mathcal{A}_{11}$, $A_{12} \in \mathcal{A}_{12}$, $A_{21} \in \mathcal{A}_{21}$, $A_{22} \in \mathcal{A}_{22}$ we have*
$\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})$.
Proof. we show that for T in $\mathcal{A}$ the following holds
$T = \Phi(A_{11} + A_{12} + A_{21} + A_{22}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21} - \Phi(A_{22}) = 0$ (2.3)
From Claim 3 we can rewrite
$(A_{11} + A_{12} + A_{21} + A_{22}) \diamond \Phi(P_1)$
$+\Phi(A_{11} + A_{12} + A_{21} + A_{22}) \diamond p_1$
$= (\Phi(A_{11} + A_{12} + A_{21} + A_{22}) \diamond p_1)$
$= \Phi((A_{11} + A_{12} + A_{21}) \diamond P_1) + \Phi(A_{22} \diamond P_1)$
$= \Phi(A_{11} \diamond P_1) + \Phi(A_{12} \diamond P_1) + \Phi(A_{21} \diamond P_1) + \Phi(A_{22} \diamond P_1)$
$= (A_{11} + A_{12} + A_{21} + A_{22}) \diamond \Phi(P_1)$
$+(\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})) \diamond P_1$
then we have
Thus, $(1 + \lambda)T_{11} + T_{12} + \lambda T_{21} = 0$ therefor$T_{11} = T_{12} = T_{21} = 0$.
similary we can show that $T_{22} = 0$ □

Claim 5. *For each $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ such that $i \neq j$, we have*
$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij})$.
Proof. for $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ we have
$(A_{ij} + P_i) \diamond (P_j + B^*_{ij}) = A_{ij} + B_{ij} + \lambda B^*_{ij} A_{ij} + \lambda B_{ij}$ (2.4)

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From equition (4) and clain (4) we have

$\Phi(A_{ij}+B_{ij})+\Phi(\lambda B^*_{ij}A_{ij})+\Phi(\lambda B^*_{ij})$

$=\Phi((A_{ij}+P_i)\diamond(P_j+B^*_{ij}))$

$=\Phi(A_{ij}+P_i)\diamond(P_j+B^*_{ij})+(A_{ij}+P_i)\diamond\Phi(P_j+B^*_{ij})$

$=(\Phi(A_{ij})+\Phi(P_i))\diamond(P_j+B^*_{ij})+(A_{ij}+P_i)\diamond(\Phi(P_j)+\Phi(B^*_{ij}))$

$=\Phi(A_{ij}\diamond P_j)+\Phi(A_{ij}\diamond B^*_{ij})+\Phi(P_i\diamond P_j)+\Phi(P_i\diamond B^*_{ij})$

$=\Phi(A_{ij})+\Phi(B_{ij})+\Phi(\lambda B^*_{ij}A_{ij})+\Phi(\lambda B^*_{ij})$.

thus

$\Phi(A_{ij}+B_{ij})=\Phi(A_{ij})+\Phi(B_{ij})$.

□

Claim 6. *For $A_{ii}, B_{ii} \in A_{ii}$ such that $1\le i\le 2$, we have*
$\Phi(A_{ii}+B_{ii})=\Phi(A_{ii})+\Phi(B_{ii})$.

Proof. We show that
$T=\Phi(A_{ii}+B_{ii})-\Phi(A_{ii})-\Phi(B_{ii})=0$.
we can write for $i \neq j$

$(A_{ii}+B_{ii})\diamond\Phi(P_j)+\Phi(A_{ii}+B_{ii})\diamond P_j$

$=\Phi((A_{ii}+B_{ii})\diamond P_j)$

$=\Phi(A_{ii}\diamond P_j)+\Phi(B_{ii}\diamond P_j)$

$=(A_{ii}+B_{ii})\diamond\Phi(P_j)+(\Phi(A_{ii})+\Phi(B_{ii})\diamond P_j)$.

therefore

$T_{ij}+\lambda T_{ji}+(1+\lambda)T_{jj}=0$. it follows that $T_{ij}=T_{ji}=T_{jj}=0$ from claim (5) every $C_{ij}\in A_{ij}$ we have

$C_{ij}\diamond\Phi(A_{ii}+B_{ii})+\Phi(C_{ij})\diamond(A_{ii}+B_{ii})$

$=\Phi(C_{ij}\diamond(A_{ii}+B_{ii}))$

$=\Phi(C_{ij}\diamond A_{ii})+\Phi(C_{ij}\diamond B_{ii})$

$=C_{ij}\diamond(\Phi(A_{ii})+\Phi(B_{ii}))+\Phi(C_{ij})\diamond(A_{ii}+B_{ii})$.

Thus,

$C_{ij}\diamond T=0$.

By primeness since $T=T_{11}+T_{12}+T_{21}+T_{22}$ we obtian $T_{ii}=0$ Hence the aditivith of Φ comes from claims (1)-(6) □

In the remainder of the paper we show that Φ is a $*-$derivation.

Theorem 2.2. *let A be a prime $*-$algebra. let the map*
$\Phi : A \to A$
satisfy the condition
$\Phi(A\diamond B)=\Phi(A)\diamond B+A\diamond\Phi(B)$ (2.5)
where $A\cdot B=A^*B-\lambda BA^*$ for $A,B\in A$ if $\Phi(I)$ and $\Phi(iI)$ are selfadjoint, then
Φ is a $*-$derivation.

Proof. we present the proof of the above theorem several claims. from theorem 2.1 we need to prove that Φ is selfadjoint and has the derivation property. □

Claim 7. *if $\Phi(I)$ and $\Phi(iI)$ are selfadjoind then $\Phi(I)=\Phi(iI)=0$*

Proof. we have
$\Phi(I\diamond I)=\Phi((1+\lambda)I)=\Phi(I)^*+\lambda\Phi(I)+(1+\lambda)\Phi(I)=2\Phi(I)+2\lambda\Phi(I)$.
thus
$\Phi(\lambda I)=\Phi(I)+2\lambda\Phi(I)$. (2.6)
on the other hand we have
$\Phi(iI\diamond iI)=\Phi((1-\lambda)I)=i(-1+\lambda)\Phi(iI)+i\Phi(iI)^*+\lambda i\Phi(iI)=2\lambda i\Phi(iI)$
thus
$\Phi(I)-\Phi(\lambda I)=2\lambda i\Phi(iI)$ (2.7)
From (2.6) and (2.7) we obtain
$-\lambda\Phi(\boldsymbol{I})=\lambda i\Phi(i\boldsymbol{I})$ (2.8)
From (2.8) we have

Thus

$(-\lambda\Phi(\boldsymbol{I}))^*=(\lambda i\Phi(i\boldsymbol{I}))^*$ (2.9)
$-\lambda\Phi(\boldsymbol{I}))=-\lambda i\Phi(i\boldsymbol{I})$ (2.10)
From (2.8) and (2.10) we hav

$\Phi(I)=\Phi(iI)=0$

□

Claim 8. *we prove that Φ preserves the star Proof.* for every $A\in A$ we have
$\Phi((1+\lambda)A)=\Phi(A\diamond I)=\Phi(A)\diamond I=(1+\lambda)\Phi(A)$
It follows that

$\Phi(\lambda A)=\lambda\Phi(A)$ (2.11)
also Thus

$\Phi(A^*+\lambda A)=\Phi(I\diamond A)=I\diamond\Phi(A)=\Phi(A)^*+\lambda\Phi(A)$
$\Phi(A^*)=\Phi(\lambda A)=\Phi(A)^*+\Phi(\lambda A)$ (2.12)
From (2.11) and (2.12) we obtian
$\Phi(A^*)=\Phi(A)^*$

□

Claim 9. $\Phi(iA)=i\Phi(A)$ *For every $A\in A$.*

Proof. for every $A\in A$ from claim (8) we have

$(-1+\lambda)\Phi(iA) = -\Phi(iA)+\lambda\Phi(iA)$

$= -\Phi(iA)+\Phi(i\lambda A)$

$=\Phi(A\diamond iI)$

$= \Phi(A)\diamond iI$

$= -i\Phi(A)+\lambda i\Phi(A)$

$=(-1+\lambda)i\Phi(A)$

Thus

From (2.13) we have
$(-1+\lambda)\Phi(iA)=(-1+\lambda)i\Phi(A)$ (2.13)
$\Phi(iA)=i\Phi(A)$

□

Claim 10. *Φ is derivation.*

Proof. For every $A,B\in A$ we have

$\Phi(AB+\lambda B^*A) = \Phi(A\diamond B^*)$

$= \Phi(A)\diamond B^*+A\diamond\Phi(B^*)$

$= \Phi(A)B+\lambda B^*\Phi(A)+A\Phi(B^*)^*+\lambda\Phi(B^*)A$.

Thus

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$\Phi(AB) + \Phi(\lambda B^*A) = \Phi(A)B + A\Phi(B) + \lambda\Phi(B^*)A$ (2.14)

on the other hand

$\Phi(AB - \lambda B^*A) = \Phi(iA) \diamond \Phi(iB^*)$

$= \Phi(iA) \diamond (iB^*) + (iA) \diamond \Phi(iB^*)$

$= -i\Phi(iA)B + i\lambda B^*\Phi(iA) + (iA)\Phi(iB^*)^* + \lambda\Phi(iB^*)(iA)$

$= \Phi(A)B - \lambda B^*\Phi(A) + A\Phi(B) - \lambda\Phi(B^*)A$

Thus

$\Phi(AB - \lambda B^*A) = \Phi(A)B + A\Phi(B) - \lambda B^*\Phi(A) - \lambda\Phi(B)^*A$ (2.15)

From (2.14) and (2.15) we have

$\Phi(AB) = A\Phi(B) + \Phi(A)B.$

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