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Analytical Solution of Biological Population of Fractional Differential Equations by Reconstruction of Variational Iteration Method

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ABSTRACT

This article presents a brand-new approximation analytical technique we refer to as the reconstruction of variational iteration method. For the goal of solving fractional biological population option pricing equations, this methodology was created. In certain circumstances, you may actually use the well-known Mittag-Leffer function to get an explicit response. The usage of the three examples below demonstrates the precision and effectiveness of the suggested method. The results show that the RVIM is not only quite straightforward but also very successful at resolving non-linear problems.

Keywords- Biological Population, Variational Iteration Method, Differential Equations.

I. INTRODUCTION

Over the last thirty years or more, fractional differential equations have grown in significance and appeal, mostly as a result of their multiple, apparently unrelated applications in the disciplines of science and engineering. For instance, the fluid-dynamic traffic model using fractional derivatives may address the inadequacy resulting from the assumption of continuous traffic flow. The nonlinear oscillation of earthquake can also be described with fractional derivatives. Additionally, many chemical processes, mathematical biology, and several other physics and engineering issues are modeled using fractional differential equations,[1]–[10].

Since most physical systems are nonlinear in nature, nonlinear issues are crucial for engineers, physicists, and mathematicians. The nonlinear equations, on the other hand, are challenging to solve and produce intriguing phenomena, such as chaos. In order to fully understand nonlinear physical events, it is crucial to investigate the precise solutions of nonlinear evolution equations. Recently, a wide variety of alternative techniques have been utilized to solve physical-interesting nonlinear and linear differential equations. Linear and nonlinear problems have been solved using the Adomian decomposition method (ADM) [11], [12], the homotopy perturbation method (HPM) [13]-[16], the variational iteration method (VIM)[16]-[21], and other techniques. Due to the challenges posed by the nonlinear variables, the Laplace transform is completely incapable of addressing nonlinear equations. Numerous strategies, including the Laplace decomposition method (LDM) [22]–[26] and the homotopy perturbation transform technique (HPTM) [27], have been put forward lately to cope with these non-linearities. A very efficient approach known as the homotopy analysis transform method (VIM) has just recently been developed by combining the homotopy analysis method (HAM) with the well-known Laplace transform [28], [29]. The variatoinal iteration technique (VIM) is used in this study to handle a variety of nonlinear issues.

The nonlinear fractional-order biological population model using the following formula is examined in this paper:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 (u^2)}{\partial x^2} + \frac{\partial^2 (u^2)}{\partial y^2} + f(u) \qquad \ldots(1)$$

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with the given initial condition

$$u(x.y.0) = f_0(x.y)$$

where u stands for population density and f for the supply of people as a result of births and deaths. This nonlinear fractional biological population model is created by substituting a fractional derivative of order with 011 $ for the first time derivative term in the associated biological population model. The derivatives are interpreted in the sense of Caputo. A parameter indicating the order of the fractional derivative is included in the general response expression and may be changed to provide different replies. The standard biological population model replaces the fractional biological population model when = 1 occurs. Other scholars have already investigated certain features of this concept [30].

In this research, we also solve the fractional biological population models using the Reconstruction of Variational Iteration Method (RVIM). To solve nonlinear fractional biological population models, the current work aims to adapt the Variational Iteration Method (VIM).

1.1. Preliminaries and definitions

In this section, we present some basic definitions and preliminaries in fractional calculus, Riemann-Liouville fractional integral of order α.

$$_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad \ldots(2)$$
$$_0D_t^{0} f(t) = f(t),$$

one useful function for fractional calculus is Mittag-Leffler function. the standard definition of te Mittag-Leffler function $E_{\alpha,\beta}(z)$ is as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \quad ,$$
$$\alpha, \beta \in \mathbb{C}, \mathrm{Re}(\alpha) > 0. \quad \ldots(3)$$

Although there are numerous ways to define fractional derivatives, in this study the most favorable definition is Capotu fraction from the order α is defined as follows.

$$_0\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau \quad \ldots(4)$$

The fractional integral of order α of function $f(t) = (t-a)^{v}$ is as follows

$$_aD_t^{-\alpha}((t-a)^{v}) = \frac{\Gamma(1+v)}{\Gamma(1+v+\alpha)}(t-v)^{v+\alpha} \quad \ldots(5)$$

II. RECONSTRUCTION OF VARIATIONAL ITERATION METHOD

In this section we introduce a approximate analytical method to solve Biological population model (1) for fractional order α $(1 \le \alpha \le 2)$.

Hesameddini and Latifzadeh [30] presented the Reconstruction of Variational Iteration Method (RVIM) for differential equations of integer order. Here, we expand this approach to solving (1) Consider the biological population equation in its generic version, which looks like this.

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} = g\left(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x \partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right),$$
$$u(x,y,0) = f_0(x,y) \quad \ldots(6)$$

where the operator $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the Caputo fractional derivatives and $m-1 \le \alpha < m$. By taking Laplas Transform from both side of equation (6) , with respect to the independent variable t and using the homogeneous initial condition, we get

$$s^{\alpha}\mathcal{L}\{u(x,y,t)\} - s^{\alpha-1}u(x,y,0)$$
$$= \mathcal{L}\left\{g\left(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x \partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right)\right\}$$

Therefore

$$\mathcal{L}\{u(x,y,t)\} = \frac{1}{s}f_0(x,y,t) +$$
$$\frac{1}{s^{\alpha}}G\left(s,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x \partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right) \quad \ldots(7)$$

Now by applying the inverse Laplace transform to both side of equation (7), and using the convolution theorem we get

$$\begin{aligned} u(x,y,t) &= f_0(x,y,t) + \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}G\left(s,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x \partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right)\right\} \\ &= f_0(x,y,t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} * g\left(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x \partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right) \\ &= f_0(x,y,t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1} g\left(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x \partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right)\mathrm{d}\xi \end{aligned}$$

according to [16] by imposing to initial condition to obtain the solution of equation (6) , we construct an iteration formula as follows

$$u_{n+1}(x,y,t) = f_0(x,y,t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1} g\left(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x \partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right)\mathrm{d}\xi \quad \ldots 8$$

where $f(0,x,y,t)$is initial solution. By the above iteration each term will be determined by the previous term in the approximation of iteration formula can be entirely evaluated. Consequently the solution may be written as

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$$u(x,y,t)=\lim_{n\to\infty}u_n(x,y,t).$$

III. EXAMPLES

Here, we apply the suggested approach to a few biological population models. The Mittag-Leffler function emerges in the resolution of these situations, as we will see.

Example 3.1:

Consider the following fractional Biological population option pricing equations.

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}+u(1-ru),\quad \ldots(9)$$

with the initial condition

$$u(x,y,0)=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right].\qquad \ldots(10)$$

Applying the RVIM method to this problem we option the following recursive formula

$$u_{n+1}(x,y,t)=f_0(x,y,t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}g\left(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right)d\xi$$

…(11)

where

$$g\left(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}+u(1-ru),$$

$$f_0(x,y,t)=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]$$

Now, the above successive approximation yields

$$u_{n+1}(x,y,t)=f_0(x,y,t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}\left(\frac{\partial^2(u_n^2)}{\partial x^2}+\frac{\partial^2(u_n^2)}{\partial y^2}+(u_n-ru_n^2)\right)d\xi$$

$$u_1(x,y,t)=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}\left(r\exp\left[\sqrt{\frac{r}{2}}(x+y)\right]+\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]-r\exp\left[\sqrt{\frac{r}{2}}(x+y)\right]\right)d\xi$$

$$\begin{aligned}u_1(x,y,t)&=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]d\xi\\&=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]+\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}d\xi\\&=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]+\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]\frac{t^\alpha}{\Gamma(\alpha+1)}\\&=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]\left[1+\frac{t^\alpha}{\Gamma(\alpha+1)}\right]\end{aligned}$$

and

$$\begin{aligned}u_2(x,y,t)&=u_1(x,y,t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}\left(\frac{\partial^2(u_1^2)}{\partial x^2}+\frac{\partial^2(u_1^2)}{\partial y^2}+(u_1-ru_1^2)\right)d\xi\\&=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]\left[1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(\alpha+1)}\right]\end{aligned}$$

finally we get

$$u_n(x,y,t)=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]\left[1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(\alpha+1)}+\cdots+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right]$$

Therefore by using the definition of Mittag-leffler function in one parameter, the solution of the problem is given by

$$\begin{aligned}u(x,y,t)&=\lim_{n\to\infty}u_n(x,y,t)=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]\sum_{m=0}^{\infty}\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\\&=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]E_{\alpha(t^\alpha)}\end{aligned}$$

if we put $\alpha=1$ we option the exact solution

$$u(x,y,t)=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right]e^t=\exp\left[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)+t\right]$$

which is an exact solution of the given classical Biological Population equation (9).

Example 3.2:

Consider the following generalized biological population model:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}+ku,\qquad \ldots(12)$$

with the initial condition

$$u(x,y,0)=\sqrt{xy},\qquad \ldots(13)$$

as in previous example we apply the RVIM method to this problem. corresponding to equation (11) recursive formula is obtained as follows:

$$u_{n+1}(x,y,t)=f_0(x,y,t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}g\left(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right)d\xi$$

where

$$\begin{aligned}g\left(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right)&=\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}+ku.\\u_0(x,y,t)&=\sqrt{x,y}\end{aligned}$$

The approximation are obtained as

$$\begin{aligned}u_1(x,y,t)&=k\sqrt{x,y}\frac{t^\alpha}{\Gamma(\alpha+1)}\\u_2(x,y,t)&=k\sqrt{xy}\frac{t^\alpha}{\Gamma(\alpha+1)}+k^2\sqrt{xy}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\u_3(x,y,t)&=k\sqrt{xy}\frac{t^\alpha}{\Gamma(\alpha+1)}+k^2\sqrt{xy}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+k^3\sqrt{xy}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\&\vdots\\u_n(x,y,t)&=k\sqrt{xy}\frac{t^\alpha}{\Gamma(\alpha+1)}+k^2\sqrt{xy}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+k^3\sqrt{xy}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots+k^n\sqrt{xy}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\end{aligned}$$

and so on.

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$$u(x,y,t)=\lim_{n\to\infty}u_n(x,y,t)=\sqrt{xy}\sum_{m=0}^{\infty}\frac{(kt^{\alpha})^m}{\Gamma(m\alpha+1)}\sqrt{xy}E_\alpha(kt^\alpha).\quad \ldots(14)$$

if we put $\alpha=1$, we obtain the exact solution:

$$u(x,y,t)=\sqrt{xy}e^{kt},\qquad \ldots(15)$$

Example 3.3:

Consider the following generalized biological population model:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}+u,\ \ldots(16)$$

with the initial condition

$$u(x,y,0)=\sqrt{\sin x \sinh y}.\ \ldots(17)$$

By applying the RVIM method to this problem. corresponding to recursive equation (16)

$$u_{n+1}(x,y,t)=f_0(x,y,t)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}g\left(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right)d\xi$$

where

$$g\left(t,x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}+u,$$
$$u(x,y,0)=\sqrt{\sin x\sinh y}.$$

Now carry out the recursive process (16) and by simplification we obtain

$$\begin{aligned}
u_1(x,y,t)&=\sqrt{\sin x\sinh y}+\frac{1}{\Gamma}\int_0^t (t-\xi)^{\alpha-1}\left(\frac{\partial^2(u_0^2)}{\partial x^2}+\frac{\partial^2(u_0^2)}{\partial y^2}+u_0\right)\\
u_1(x,y,t)&=\sqrt{\sin x\sinh y}\left(1+\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\\
u_2(x,y,t)&=\sqrt{\sin x\sinh y}\left(1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right)\\
&\vdots\\
u_n(x,y,t)&=\sqrt{\sin x\sinh y}\left(1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right)\\
u(x,y,t)&=\lim_{n\to\infty}u_n(x,y,t)=\sqrt{\sin x\sinh y}\sum_{m=0}^{\infty}\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\\
&=\sqrt{\sin x\sinh y}E_\alpha(t^\alpha)
\end{aligned}$$

if we put $\alpha=1$, we the exact solution

$$u(x,y,t)=\sqrt{\sin x\sinh y}e^t,\qquad \ldots(18)$$

IV. CONCLUSION

Three examples of population equations used in option pricing are provided in this article. The (RVIM) is successfully used in these cases. In the recursive process, the Mittag-Leffler function always arises, and the closed form of solutions is obtained. The findings shown in [11], [31], and [32] are consistent with the depicted outcomes for two of the situations. at least as far as we are aware. However, as we could see, it may be effectively addressed utilizing (RVIM). As a result, the (RVIM) approach is effective for locating the solutions to fractional partial differential equations. Additionally, the series solution is often simple to discover. Only a handful of the series' keywords need to be found; the rest will be figured out on their own.

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