

Amazement of Complex Integration

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ABSTRACT

In this article, we discuss the facts and wonders of complex integrations. We describe the differences between integrations and complex integrations. These differences show the wonders of complex integrations.

Keywords- complex integration, analytic function, smooth curve, Residue, Laurent series, integrable, derivative.

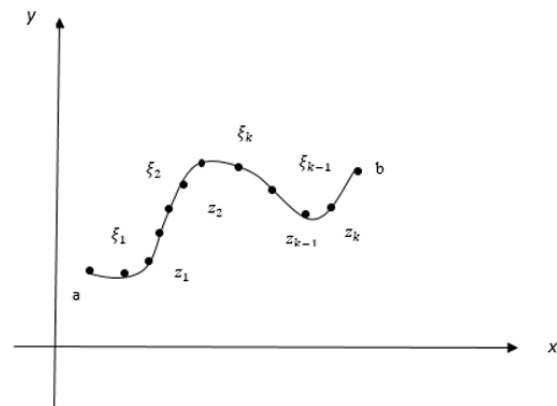
I. INTRODUCTION

In general, we have no rules (techniques) to integrate the complex functions regardless of exceptions. As we know, there are many rules for the integration of real functions, even so, they are not applicable in the field of complex functions, and there aren't any special rules. Therefore, we aim to use the techniques that describe the value of the given integral is equal or less than of some quantity, but sometimes it is not easy to use the techniques or change the form of the questions for conformity to the relevant criteria because it is essential to know many other conditions and criteria. This article examines the wonders of mixed integrals, which are effective in integrating, and the tricks we use instead of integrating methods.

II. BACKGROUND

Integrating on a path: Integration of complex-valued functions of a complex variable are defined on curves in the complex plane, rather than on just intervals of the real line. Classes of curves that are adequate for the study of such integrals are introduced in this article.

Let $x(t)$ and $y(t)$ be continuous real-valued functions of a real variable t in $[a, b]$, Assume that C is a smooth curve with equation $(C): z(t) = x(t) + iy(t)$, $t \in [a, b]$



If $f(z)$ be a continuous function on C , in the integral $[a, b]$, then the desired integral can be set to points $a = t_0, t_1, \dots, t_n = b$ where $t_0 < t_1 < \dots < t_n$, and this causes the C curve to be presented to z_0, z_1, \dots, z_n where $z_i = z(t_i)$. Suppose $\Delta z_i = z_i - z_{i-1}$ and ξ_i be the desired point between z_i and z_{i-1} , then the sum will be:

$$S_n = \sum_{k=1}^n f(\xi_k) \Delta z_k$$

Let the number of subdivisions n increase in such a way that the largest of the chord lengths $|\Delta z_k|$

approaches zero. Then, since $f(z)$ is continuous the sum s_n approaches a limit which does not depend on the mode of subdivision and we display this limit like:

$$\int_a^b f(z)dz \text{ or } \int_c f(z)dz$$

Which called the complex line integral or simply line integral of $f(z)$ along the directional curve C, or the definite integral of $f(z)$ from a to b along curve C. in such case, $f(z)$ is said to be integrable along C. if $f(z)$ is analytic at all points of a region \mathfrak{R} and if C is a curve lying in \mathfrak{R} , then $f(z)$ is continuous and therefore integrable along with C. [11]

III. LAURENT SERIES

Suppose that the function $f(z)$ across the arc area $R_1 < |z - z_0| < R_2$, with the analytical center z_0 , and C represents the simple and closed path (Jordan curve) in a positive direction around z_0 , located in this area (as shown in Figure In this case, at any point z of that area, f (z) has a series representation. [10]

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (R < |z - z_0| < R_2)$$

Where:

$$a_n = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 0,1,2, \dots) , b_n = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{(z - z_0)^{-n+1}} \quad (n = 1,2, \dots)$$

Topic Analysis: It is important to understand the techniques of integrations of real-valued functions although they are not applicable for complex integrations. The issue that makes complex integration more complicated is that we don't know what we get by integrating from a curve or integrating from a shape. The length of a curve, area of a plane, and volumes can be obtained by definite real integrations and indefinite integration is studied as the inverse of derivatives, which allow us to guess the value of integral, but the complex integration is neither inverse of derivative nor calculation of the length of curve, area or volume.

Let start by describing the following theorem

Theorem: if G be a region contained a smooth curve C and $f(z)$ be a continuous function on G, then $f(z)$ is integrable on whole C. [9]

Theorem: if w(t) is a piecewise continuous complex-valued function defined on an interval $a \leq \theta \leq b$, [10] then

$$\left| \int_a^b w(t)dt \right| \leq \int_a^b |w(t)| dt \quad (a \leq b)$$

As an example, let C be right half of a circle $|z| = 2$ from $z = -2i$ to $z = 2i$. Calculate the value of integral $I = \int_c \bar{z}dz$, $z = 2e^{i\theta}$ ($-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$).

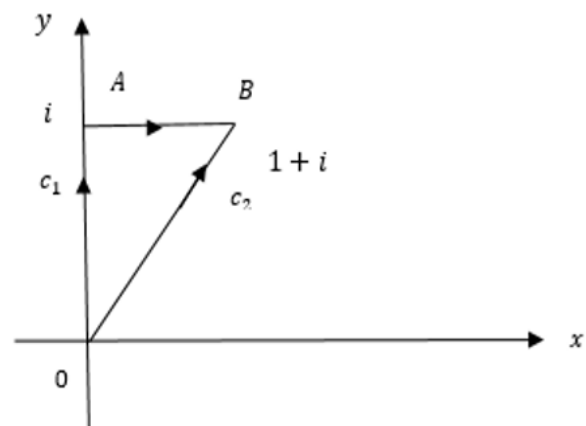
So

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overline{2e^{i\theta}} (2e^{i\theta})' d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2e^{-i\theta} 2ie^{i\theta} d\theta = 4i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = 4\pi i$$

Here, the impressive points are, the calculation of integral is definite, integration is on a circle, and integration is on the circle bounded from $z = -2i$ to $z = 2i$. otherwise, there is no physical or geometrical analysis. [12]

The interesting point at the figure below is that we get different results of moving from O to B, but according to vector analysis, the direction c_1 and c_2 must be the same. Therefore, the path is important in complex integrations.

First, calculate the integral on c_1 which is the curve OAB



$$\int_{c_1} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz, \quad \int_{c_2} f(z)dz = \int_{OB} f(z)dz$$

The physical and geometrical analysis of the differential and integral calculus is that these two paths must have the same result, but in practice, it seems that these two integrals are not the same. It means $\int_{c_1} f(z)dz \neq \int_{c_2} f(z)dz$ and the function is: [10]

$$f(z) = y - x - i3x^2 \quad (z = x + iy).$$

$$\int_{OA} f(z)dz = \int_0^1 yidy = i \int_0^1 ydy = i \frac{1}{2}$$

$$\int_{AB} f(z)dz = \int_0^1 (1-x-i3x^2) 1dx = \int_0^1 (1-x) dx - 3i \int_0^1 x^2 dx = \frac{1}{2} - i$$

$$\int_{c_2} f(z)dz = \int_0^1 -i3x^2(1+i)dx = 3(1-i) \int_0^1 x^2 dx = 1 - i$$

Even though started points and ended points of both directions are the same. The value of integral of $f(z)$ in one direction is different with the other. Therefore the value of integral $f(z)$ on direction OABO or $c_1 - c_2$ is not equal to zero:

$$\int_{c_1} f(z)dz - \int_{c_2} f(z)dz = \frac{-1+i}{2}$$

Since we got different results, it may imply that the calculation purpose is obtaining the length of the curve, but we will see that the calculation of integral is dependent on the initial point and terminal point not to the direction of the path. As c_1 and c_2 have the same initial point and terminal point, and the result of integral are different it shows that integration is related to the path. But see the problem below. Here, we consider that C is an arbitrary smooth curve such as $z = z(t)$, ($a \leq t \leq b$) at initial point $z(a) = z_1$ to terminal point $z(b) = z_2$, to calculate the value of integral:

$$I = \int_{c_2} f(z)dz = \int_a^b z(t)z'(t)dt = \frac{[z(t)]^2}{2} \Big|_a^b = \frac{[z(b)]^2 - [z(a)]^2}{2} = \frac{z_2^2 - z_1^2}{2}$$

It is obvious that the integration of real-valued functions is not independent of the form of the curve, but in complex integrations, especially the above relation, it depends on the initial point and terminal point and independent from the form of the curve. In the previous example, we observed that the result of integration depended on the form of the curve and independent of initial and terminal points. Although the initial and terminal points were the same, the values of the integration of various paths were different. [5]

To generalize the issue for uneven curves, we can divide the curve C to the limit number of smooth curves $c_k (k = 1, 2, \dots, n)$ which ended point of one is the started point of the other. In other words, we assume that the curve c_k is continued from z_k to z_{k+1} . We can write

$$\int_C z dz = \sum_{k=1}^n \int_{c_k} z dz = \sum \frac{z_{k+1}^2 - z_k^2}{2} = \frac{z_{n+1}^2 - z_1^2}{2}$$

As a result, if C is a piecewise smooth curve such that connects two points z_0 and z , then we have

$$\int_C z^n dz = \frac{1}{n+1} (z^{n+1} - z_0^{n+1})$$

Therefore, if C is a closed curve, the value of integration is zero because here we have $z^{n+1} = z_0^{n+1}$. In the case of integration, if we are dealing with a curve that is not closed but smooth, then it is possible to integrate it simply, but for the uneven curve, we have to partition the curve to the smooth curves $c_k (k = 1, 2, \dots, n)$ and integrate from each curve and the summation of values is same as the value of integral C . As a result, the integration of $f(z) = z$ is on the closed curve is zero (compare to one of the previous examples that the integration on the closed curve was not zero) [2]

Theorem: let $f(z)$ be a continuous function on smooth curve C which is contained on region G , then $f(z)$ is integrable on the length C .

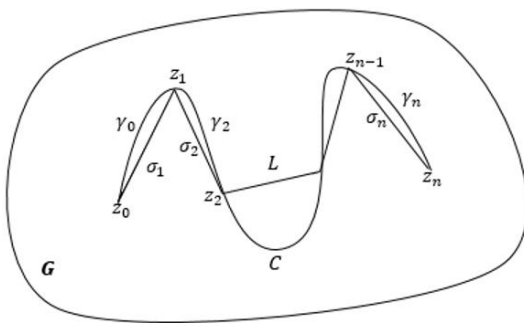
Theorem: Suppose $f(z)$ is integrable along a curve C having finite length L and suppose there exists a positive number M such that $|f(z)| \leq M$ on C then $|\int_C f(z)dz| \leq ML$. [12]

The theorem does not say anything about the exact value of the integral, but it helps us to recognize the value of the integrals which are less than or equal to some real or complex values. if we compare this method

with methods on the integration of real-valued functions, this would not be an appropriate method but in complex integration, this is valuable because here we have no formula to calculate the integral of a complex function. The method is useful not only for calculation of the approximate value of the integral of some function but also is useful to prove many theorems of complex integration. The theorem paves the way to prove the related theorem and to realize the related problems.

Theorem: Let $f(z)$ be a continuous function on piecewise smooth curve C which is contained on region G , then for every $\varepsilon > 0$ there is a polygon in C which is surrounded by L such that

$$\left| \int_C f(z) dz - \int_L f(z) dz \right| < \varepsilon$$

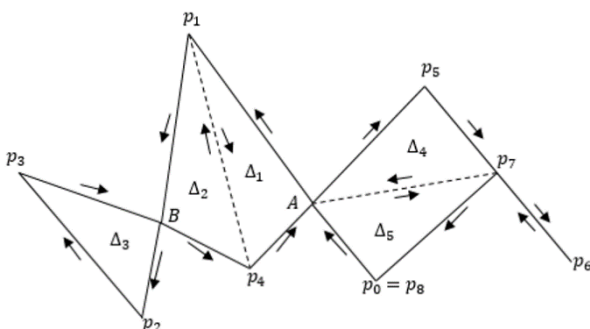


We cannot integrate $f(z)$ on curve C , we will be able to cover the curve C by another curve like L such as the curve C must be located on the same region and also $f(z)$ be continuous on the piecewise smooth curve L , by doing this we are assured that the integral of the function $f(z)$ is the same as integral on C . [3]

Theorem: let the function $f(z)$ be continuous on the connected and simple region G which contained the polygon L then [4]

$$\int_L f(z) dz = \int_{\Delta_1} f(z) dz + \int_{\Delta_2} f(z) dz + \dots + \int_{\Delta_n} f(z) dz$$

Here, the curves $\Delta_1, \Delta_2, \dots, \Delta_n$ are the borders of triangles which are located on the region G

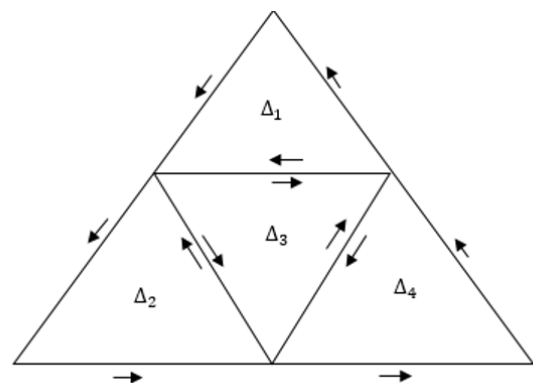


Closed polygonal shapes are interchangeable to triangles. We divide them into triangles and apply integration on every triangle and collect the values of integral then the result would be the same as to the integral of a function on a region closed polygonal shapes. When we divide the shape into triangles and integrate from every side of the triangle. We may think that two times integration from the same sides of triangles will arise the value of integral much bigger, but fortunately, our assumption is not true. It is interesting to know that the integral value of those sides which are common between to triangles will be eliminated because it would be integrated twice with the opposite directions and integral of the same side with opposite direction eliminate each other. In this manner we will get the actual value of the integral on curve C . here the partition of the shape in triangles let us calculate the integral without leaving a footprint, this trick is as good as the catalyst in chemistry (a substance that increases the rate of a chemical reaction without itself undergoing any permanent chemical change).

When we limit the region inside of another connected simple closed region G such that the function $f(z)$ which is continuous on closed curve L , we apply the same method in definite integral in real-valued functions although these two are not comparable because if we divide any area in some shapes then integrate every shape and collect the values, In fact, we collect the areas, but here if we ask, do we get the area in this method in complex integration? Unfortunately, we have no answer to this question.

Theorem: If a function $f(z)$ is analytic at all points interior to and a simple closed contour C , then [11]

$$\int_C f(z) dz = 0$$



The theorem is proved by using triangles such that the region G supposed to be a triangle and the triangle divided into three triangles and continued like that up to n , now we can conclude that in a region which has Regular geometric shape we can apply this method and calculate the value of integral. Cauchy integral's

formula led us to any closed curve which satisfies the conditions of the theorem instead of searching any other methods. we can apply the theorem and in this case, the value of the integral is always zero.

The generalization of the Cauchy integral formula shows that if the function $f(z)$ is analytic in a region I and just continuous in \bar{I} , the value of integral will be again zero [7]

$$\int_C f(z) dz = 0$$

This is known as the generalization of Cauchy integral formula.

Theorem: The advantage of this theorem is that we will be able to consider the region as an Irregular shape and all the curves that are located inside of the region D can also be an Irregular shape, the only limitation is that the curves must not intersect and overlap each, and C_0 contain all the curves C_1, C_2, \dots, C_n where C_0 and C_1, C_2, \dots, C_n are Jordan and piecewise smooth curves and also the set of the curve which are inside of the C_0 makes a $(n + 1)$ connected domain such that the border of the region is supposed to be the borders of $n+1$ Jordan curves. Assume that the functions $f(z)$ is analytic on \bar{D} , by theorem we can take the irregular shapes D and divide it to the sub regions C_1, C_2, \dots, C_n and calculate the integral, this method can help us to calculate the integral of irregular shapes. [6]

$$\int_{C_0} f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz + \dots + \int_{C_n} f(z) dz$$

Cauchy's Integral Formula: If we present the form of the related functions as $\int_C \frac{f(z)}{z-z_0} dz$ we have come up with the best possible answer, which means that we can claim that the value of integral is $2\pi i$, it means $\int_C \frac{f(z)}{z-z_0} dz = 2\pi i$. we assume that the function $f(z)$ is analytic on region G which contains the Jordan, closed and piecewise smooth curve C and z_0 is also included. [10]

Alongside this, it relates the values of $f(z)$ inside C to the values of $f(z)$ on the C curve. Note that, if z_0 is outside C, then $\frac{f(z)}{z-z_0}$ is analytical in and around c and therefore according to Cauchy's integral theorem. we have $\frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz = 0$.

Now, another interesting point is that if we assume (Mora's theorem) that the function $f(z)$ in the G region is continuous and the curve C of the Jordan curve is a smooth part, and also the function $f(z)$, in all sub-regions, the closed part is a smooth. C located in G has properties $\int_{C_i} f(z) dz = 0$ In this case, the function $f(z)$ in G is analytical. [10]

The Derivative Of Analytic Functions: We can make a conclusion from Cauchy's integral formula that if a function is analytic at a point then the derivatives of any degree exist and analytic at that point. Luckily the derivatives of analytic functions are another helpful way for calculation of complex integrals and in this method, we may often use the series like Laurent series so the first and second derivative of analytic functions are [11]

$$\int_C \frac{f(s)}{(s-z)^2} ds = f'(z)2\pi i \quad , \quad \int_C \frac{f(s)}{(s-z)^3} ds = f''(z)\pi i$$

Therefore, in general, we write

$$\int_C \frac{f(s)}{(s-z)^3} ds = 2\pi i f^{(n)}(z) \frac{1}{n!}$$

Where z is included, the curve C and S show the point, which is on the boundary, points of C.

Residues: The residue is the most powerful method for complex integration in this case, we need to know the Laurent series and based on this series we have to extend the functions as series and find b_1 and after that, we can easily calculate the required complex integral, as we know in Laurent series we have a_n and b_n which are the coefficient of the Laurent series, there is no need to introduce and calculate a_n only the formula of b_n is enough which is [10]

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z-z_0)^{-n+1}} \quad (n = 1, 2, \dots)$$

Where $n=1$ we have $b_1 = \frac{1}{2\pi i} \int_C f(z) dz$ which is the best way to calculate $\int_C f(z) dz$.

The complex number b_1 is the coefficient of $\frac{1}{z-z_0}$ in Laurent series and called the residue of the function $f(z)$ at a singular z_0 and write as $Res f(z)$.

Without any exceptional conditions that we have faced in very simple functions, we have no other way around to calculate the complex integral therefore the method of residue is very useful for complex integrations.

If we realize the method of residue we would see the attraction, the procedure of calculating $\int_C f(z) dz$ is an interesting technique. we use derivative, Laurent series, and sometimes Tylor series to calculate the integrals. Although the way is long and tough, we have to apply it because we have no other way around. [11]

For instance, we consider a circle C with $|z - 2| = 1$ and calculate the value of $\int \frac{dz}{z(z-2)^4}$

As we know here 0 and 2 are the isolated singular points of the function $f(z) = \frac{1}{z(z-2)^4}$ the

functions have Laurent series at the neighborhood $0 < |z - 2| < 2$, so we can write

$$\begin{aligned} \frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n \quad (|z| < 1) \\ \Rightarrow \frac{1}{z(z-2)^4} &= \frac{1}{(z-2)^4} \cdot \frac{1}{z} = \frac{1}{(z-2)^4} \cdot \frac{1}{2+(z-2)} \\ &= \frac{1}{(z-2)^4} \cdot \frac{1}{2\left(1+\frac{(z-2)}{2}\right)} \\ &= \frac{1}{2(z-2)^4} \cdot \frac{1}{1-\left(-\frac{(z-2)}{2}\right)} \\ &= \frac{1}{2(z-2)^4} \cdot \sum_{n=0}^{\infty} \left[\frac{-(z-2)}{2}\right]^n \\ &= \frac{1}{2(z-2)^4} \cdot \sum_{n=0}^{\infty} (-1)^n \left[\frac{(z-2)}{2}\right]^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^{n-4}}{2^{n+1}} \end{aligned}$$

Here for $n=3$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^{n-4}}{2^{n+1}} &= (-1)^3 \frac{(z-2)^{-1}}{2^4} \\ &= -\frac{1}{16(z-2)} \end{aligned}$$

Therefore $b_1 = -\frac{1}{16}$ and by using the method of residue we have

$$\int \frac{dz}{z(z-2)^4} = 2\pi i b_1 = 2\pi i \left(\frac{-1}{16}\right) = \frac{-\pi i}{8}$$

IV. CONCLUSION

By surveying and comparing the integral of real-valued functions and integral of complex-valued functions and also usage of derivatives theorems in integration we can state the results as below:

1. In general, the methods and techniques, which are used in the integration of real-valued functions are not applicable in complex integrations.

2. Complex integrations depend on techniques and theorem that help one to calculate the integral, and there is no special technique, which is directly applicable.

3. The derivatives of complex functions are not useful for integration as much as the derivatives of real-valued functions are.

4. Without some exceptional conditions, we have no physical or geometrical explanation to describe complex integrations.

5. The first and foremost method of complex integrations is residue, and mathematicians need to work hard on residue and expand it.

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