

Modified Hermite - Hadamard Inequalities: Analytical Extensions and Computational Applications

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ABSTRACT

The Hermite–Hadamard inequality is a cornerstone of convex analysis, providing a two-sided estimate that links the midpoint value, the integral average, and the endpoint average of a convex function. Over the past century, numerous modifications and refinements have been developed to extend its applicability to broader function classes, fractional integral operators, weighted settings, and numerical quadrature. This paper presents a unified analytical review of the most significant modified Hermite–Hadamard inequalities and evaluates their computational performance. The study covers generalizations based on ss-convex, h-convex, harmonically convex, quasi-convex, m-convex, and (α, m) -convex functions; fractional versions using Riemann–Liouville integrals; derivative-based error bounds; weighted and Fejér-type formulations; and interval subdivision techniques. A systematic computational benchmarking methodology is employed using polynomial, exponential, reciprocal, and radical test functions, with error metrics including bound width, relative position, and convergence under subdivision. The results show that no single modification is universally superior; rather, the choice depends on the function class, available derivative information, required accuracy, and computational cost. Practical selection guidelines are provided, and open problems are identified, including the need for unified error estimates, variable-order extensions, and multidimensional generalizations. This paper aims to serve as a comprehensive resource for researchers and practitioners interested in the theory and applications of Hermite–Hadamard-type inequalities.

Keywords- Hermite–Hadamard inequality; convexity; ss-convex functions; h-convex functions; harmonically convex functions; fractional integrals; Fejér inequality; numerical quadrature; error bounds.

I. INTRODUCTION

The Hermite–Hadamard inequality stands as one of the most fundamental and elegant results in the theory of convex functions, establishing a profound connection between the value of a convex function at the midpoint of an interval, its integral average over that interval, and the arithmetic mean of its endpoint values. For a convex

function $f: [a, b] \rightarrow \mathbb{R}$, this classical double inequality is expressed as

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (1.1)$$

Since its independent discovery by Charles Hermite and Jacques Hadamard in the late nineteenth century [1], this inequality has become a cornerstone of mathematical analysis, providing essential tools for

estimating integrals, analyzing convex functions, and establishing error bounds in numerical quadrature rules [2, 3]. The significance of (1) lies not only in its simplicity but also in its remarkable generality and the depth of its geometric interpretation: for a convex function, the graph lies below the chord connecting its endpoints, and the integral average is bounded between the midpoint value and the trapezoidal approximation [4, 5].

Over the past century, the Hermite–Hadamard inequality has undergone extensive generalization and refinement, driven by the need to address broader classes of functions, more general integral operators, and increasingly complex analytical structures arising in pure and applied mathematics. The classical form (1) assumes that f is convex in the ordinary sense, but many functions encountered in modern analysis exhibit weaker or generalized forms of convexity. This observation has motivated the development of numerous variants of (1) for functions that are ss -convex [6, 7], hh -convex [8, 9], harmonically convex [10], quasi-convex [11], mm -convex [12], and (α, m) -convex [13], among others. Each of these generalizations extends the applicability of the Hermite–Hadamard inequality to new function classes while preserving its essential structure.

Parallel to these developments in convexity theory, the emergence of fractional calculus has opened new avenues for extending the Hermite–Hadamard inequality. Fractional integrals, particularly those of Riemann–Liouville type, provide a natural framework for capturing non-local and memory-dependent phenomena that cannot be adequately described by classical integer-order integrals [14, 15]. Sarikaya et al. [16] established the fractional analogue of (1) using Riemann–Liouville fractional integrals, showing that:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \\ &\leq \frac{f(a) + f(b)}{2} \quad (1.2) \end{aligned}$$

Where $J_{a+}^\alpha f(x)$ and $J_{b-}^\alpha f(x)$ denote left- and right-sided fractional integral operators of order $\alpha > 0$. This fractional version reduces to the classical inequality when $\alpha = 1$, demonstrating the consistency of the generalization [16, 17].

Another significant line of development concerns refinements of the Hermite–Hadamard inequality through interval subdivision. By partitioning the interval $[a, b]$ into n equal subintervals and applying (1) to each subinterval, one obtains composite midpoint and trapezoidal bounds that converge to the integral average as $n \rightarrow \infty$ [3, 18]. This approach not only sharpens the inequality but also establishes a direct link between the Hermite–Hadamard inequality and composite numerical quadrature rules, thereby bridging theoretical analysis and computational practice [19, 20]. Recent work by Bouharket et al. [21] has extended this refinement strategy using symmetric subdivisions and convexity arguments, yielding hierarchical sequences of

lower and upper bounds that systematically improve upon the classical result.

The incorporation of derivative information represents yet another important refinement strategy. Dragomir and Agarwal [2] established error bounds for the trapezoidal approximation in terms of the first derivative, showing that if $|f'|$ is convex, then:

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|) \quad (1.3) \end{aligned}$$

Subsequent work by Pearce and Pečarić [22] and Kirmaci et al. [23] extended these estimates to L^q spaces and s -convex derivatives, respectively, providing progressively sharper error bounds under weaker regularity assumptions.

Weighted versions of the Hermite–Hadamard inequality, known as Hermite–Hadamard–Fejér inequalities, generalize (1) by introducing a weight function g that is symmetric about the midpoint [24, 25]:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx &\leq \int_a^b f(x) g(x) dx \\ &\leq \frac{f(a) + f(b)}{2} \int_a^b g(x) dx \quad (1.4) \end{aligned}$$

This weighted formulation is particularly useful in contexts where different points in the interval carry different importance, such as in probability theory, statistical mechanics, and signal processing [26, 27].

Despite the extensive body of literature on Hermite–Hadamard-type inequalities, several gaps remain. First, many existing generalizations and refinements are scattered across specialized journals and are not systematically compared or unified. Second, the practical implications of these modifications for computational mathematics—particularly in numerical integration, error analysis, and algorithm design—are often underemphasized. Third, the selection of an appropriate modified inequality for a given problem depends on multiple factors, including the nature of the function, the available derivative information, and the desired accuracy, yet comprehensive guidance on this selection process is lacking.

The present paper aims to address these gaps by providing a systematic analytical review of modified Hermite–Hadamard inequalities and their computational applications. Specifically, we pursue the following objectives:

To present a unified framework for understanding the classical Hermite–Hadamard inequality and its principal modifications, including those based on generalized convexity, fractional integrals, interval subdivision, derivative information, and weighted formulations.

To analyze the analytical properties of each modification, including conditions for validity, sharpness of bounds, and relationships to the classical inequality.

To evaluate the computational performance of these modifications through numerical examples involving polynomial, exponential, reciprocal, and radical functions, comparing the accuracy and efficiency of different bounds.

To provide practical guidance on the selection of appropriate modified inequalities for specific classes of functions and computational tasks.

The structure of the paper is as follows. Section 2 establishes the theoretical foundations, reviewing the classical Hermite–Hadamard inequality, the definition and properties of convex functions, and the basic analytical tools used throughout. Section 3 presents modifications based on generalized convexity, including ss-convex, hh-convex, harmonically convex, and quasi-convex functions. Section 4 develops fractional integral versions of the inequality, with emphasis on Riemann–Liouville operators and their properties. Section 5 examines refinements obtained through interval subdivision and composite bounds. Section 6 addresses derivative-based error bounds and their applications to numerical quadrature. Section 7 presents weighted and Fejér-type generalizations. Section 8 provides comprehensive numerical comparisons and computational analysis. Finally, Section 9 summarizes the main findings and outlines directions for future research.

By synthesizing theoretical developments with practical computational considerations, this paper aims to serve as a comprehensive resource for researchers, graduate students, and practitioners interested in the analytical and numerical applications of Hermite–Hadamard-type inequalities.

II. METHODOLOGY

2.1 Research Design

This study adopts a combined descriptive–analytical, comparative, and computational methodology. The analytical component is devoted to reviewing, unifying, and interpreting the principal modifications of the Hermite–Hadamard inequality. The computational component is designed to evaluate the accuracy, sharpness, and practical applicability of these modifications through selected benchmark functions and error metrics. The research is primarily library-based and theoretical; no field data or experimental measurements are used. The source material includes peer-reviewed journal articles, monographs, graduate theses, and reference works on convex analysis, integral inequalities, fractional calculus, and numerical quadrature [1–5].

2.2 Literature Selection and Classification

Relevant literature was identified using keyword searches including Hermite–Hadamard inequality, refinement, generalization, s-convex, h-convex, harmonically convex, quasi-convex, fractional integral,

Fejér inequality, and quadrature rules. Sources were included if they satisfied at least one of the following criteria:

1. they present a refinement, extension, or modification of the classical Hermite–Hadamard inequality;
2. they introduce a generalized convexity class and derive corresponding integral bounds;
3. they employ fractional integral operators in Hermite–Hadamard-type inequalities;
4. they provide derivative-based error estimates or numerical applications.

The selected works were classified along five dimensions:

- type of convexity: ordinary convex, ss-convex, h-convex, harmonically convex, quasi-convex, mm-convex, (α, m) -convex [6–13];
- integral operator: classical Riemann integral, Riemann–Liouville fractional integral, weighted integral [14–17];
- interval treatment: single interval, symmetric subdivision, composite quadrature [18–21];
- derivative information: first derivative, second derivative, LqLq bounds [2, 22, 23];
- weight function: unweighted, Fejér-type symmetric weights [24–27].

2.3 Analytical Framework

The analytical framework begins with the classical Hermite–Hadamard inequality for a convex function $f: [a, b] \rightarrow \mathbb{R}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (2.1)$$

For each modification, the following procedure is applied:

1. **State the generalized convexity condition or integral operator.**

For example, s-convexity in the second sense is defined by

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y), t \in [0, 1], s \in (0, 1] \quad (2.2)$$

leading to the modified bounds

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1} \quad (2.3)$$

[6, 7, 23].

Derive the corresponding lower and upper bounds.

For h-convex functions,

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \quad (2.4)$$

one obtains

$$\frac{1}{b-a} \int_a^b f(x) dx \leq (f(a) + f(b)) \int_0^1 h(t) dt \quad (2.5)$$

[8, 9]. For harmonically convex functions, the inequality takes the form

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a)+f(b)}{2} \quad (2.6)$$

[10]. For quasi-convex functions, the upper bound becomes

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \max\{f(a), f(b)\} \quad (2.7)$$

[11].

Extend to fractional integrals.
Using the Riemann–Liouville operators J_{a+}^{α} and J_{b-}^{α} , the fractional Hermite–Hadamard inequality is written as

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \leq \frac{f(a)+f(b)}{2} \quad (2.8)$$

for $\alpha > 0$ [16, 17].

Include weighted and Fejér-type forms.

If g is nonnegative and symmetric about the midpoint, i.e. $g(a+b-x) = g(x)$, then

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \leq \int_a^b f(x) g(x) dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x) dx \quad (2.9)$$

[24–27].

Verify reduction to the classical case.

Each modification is checked by setting the relevant parameter to its classical value: $s = 1$, $h(t) = t$, $\alpha = 1$, $g(x) = 1$, or by taking the ordinary convexity condition. Equality cases and sharpness are examined using simple functions such as $f(x) = x^2$, $f(x) = e^x$, and $f(x) = 1/x$.

2.4 Derivation and Verification Procedure

For every modified inequality, the derivation follows a consistent sequence:

State assumptions on f , the interval $[a, b]$, differentiability, positivity, and the relevant convexity or fractional parameter.

Apply the appropriate analytical tool, such as Jensen's inequality, Hölder's inequality, power-mean inequality, integration by parts, or the change of variable $x = ta + (1-t)b$.

Derive lower and upper bounds and simplify the resulting expressions.

Verify consistency by reducing the result to the classical Hermite–Hadamard inequality or to a known special case.

Test sharpness by constructing equality cases or near-equality examples. For instance, ss -convex bounds become exact for $f(x) = x^s$ on suitable intervals, while derivative-based trapezoidal bounds are exact for quadratic functions [2, 6, 23].

For derivative-based refinements, the standard identity

$$\begin{aligned} \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \\ = \frac{b-a}{2} \int_0^1 (1-2t) f'(ta) \\ + (1-t)b dt \end{aligned} \quad (2.10)$$

is used as the starting point [2]. Applying Hölder or power-mean inequalities then yields bounds involving $|f'|$ or $|f''|$, such as

$$\begin{aligned} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|) \end{aligned} \quad (2.11)$$

when $|f'|$ is convex [2, 22].

2.5 Computational Framework

The computational part of the methodology is designed to compare the sharpness and practical behavior of the modified inequalities. Four benchmark functions are used:

$$\begin{aligned} f_1(x) &= x \text{ on } [0, 2]; \\ f_2(x) &= e^x \text{ on } [0, 1]; \\ f_3(x) &= \frac{1}{x} \text{ on } [1, 4]; \\ f_4(x) &= \sqrt{x} \text{ on } [0, 4]. \end{aligned}$$

For each function, the following quantities are computed:

$$\begin{aligned} L = f\left(\frac{a+b}{2}\right), I = \frac{1}{b-a} \int_a^b f(x) dx, U \\ = \frac{f(a)+f(b)}{2} \end{aligned} \quad (2.12)$$

or the corresponding modified upper bound. To measure the quality of each inequality, the following error metrics are used:

$$\begin{aligned} E_L = I - L, \quad E_U = U - I, \quad W = U - L, R \\ = \frac{I - L}{U - L}, \quad 0 \leq R \leq 1 \end{aligned} \quad (2.13)$$

Here W is the width of the bounding interval; smaller W indicates a sharper inequality. The ratio R describes the relative position of the true integral average between the lower and upper bounds.

Composite refinements are evaluated by partitioning $[a, b]$ into n equal subintervals. The composite midpoint and trapezoidal bounds are computed as

$$\begin{aligned} M_n = \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right), T_n \\ = \frac{1}{n} \left[\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(x_i) \right] \end{aligned} \quad (2.14)$$

The values $n=1, 2, 4, 8$ are used to observe convergence toward the integral average. For derivative-

based bounds, the maximum of $|f''|$ on $[a, b]$ is estimated and compared with the exact errors

$$\frac{M(b-a)^2}{12}, \frac{M(b-a)^2}{24} \quad (2.15)$$

For fractional versions, the parameter α is varied over $\alpha = 0.5, 1, 1.5, 2$ and the effect on the bounds is recorded. For Fejér-type inequalities, symmetric weights such as $g(x) = 1$, $g(x) = x(a+b-x)$, and $g(x) = \sin(\pi(x-a)/(b-a))$ are tested.

All numerical computations are performed with high precision, and relative errors are reported to avoid scale dependence.

2.6 Comparative Analysis and Selection Criteria

The modified inequalities are compared according to four criteria:

Applicability: the class of functions for which the inequality is valid.

Sharpness: the width W of the bounding interval and the closeness of L and U to I .

Computational cost: the amount of derivative, fractional, or quadrature information required.

Parameter sensitivity: the dependence of the bounds on parameters such as s , h , α , or the weight g .

The comparison yields practical selection guidelines. Classical Hermite–Hadamard is preferred for ordinary convex functions when simplicity is sufficient. s -convex and h -convex versions are useful when the function satisfies a weaker or parameterized convexity condition [6–9, 23]. Harmonically convex bounds are natural for reciprocal and rate-type quantities [10]. Quasi-convex bounds apply to a broader class of functions but may be less sharp [11]. Fractional versions are appropriate for non-local or memory-dependent models [14–17]. Fejér-type inequalities are used when different points of the interval have different weights [24–27]. Interval subdivision is the most direct way to improve numerical accuracy [18–21], while derivative-based bounds provide explicit error control in quadrature problems [2, 22, 23].

2.7 Limitations

The methodology has several limitations. First, the study is theoretical and computational; it does not include experimental validation. Second, many modified inequalities require additional assumptions such as positivity, differentiability, or restrictions on the interval. Third, the numerical examples are illustrative rather than exhaustive, and the observed sharpness may depend on the chosen functions and parameters. Fourth, fractional and weighted versions may require numerical integration, which introduces its own approximation error. Finally, the results are not intended to cover all possible non-convex functions; rather, they provide a structured comparison of the most widely used modifications of the Hermite–Hadamard inequality.

III. PRIOR APPROACH

The classical Hermite–Hadamard inequality for a convex function $f: [a, b] \rightarrow R$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2} \quad (3.1)$$

has motivated a broad spectrum of modifications, refinements, and generalizations. These prior approaches can be grouped into several main directions: generalized convexity, fractional integrals, derivative-based error bounds, weighted and Fejér-type formulations, interval subdivision and composite rules, product-function inequalities, and non-local or variable-order extensions. This section reviews the principal contributions in each direction and identifies the gaps that motivate the present study.

3.1 Classical refinements and early extensions

The original inequality (1) was established in the late nineteenth century [1] and later systematically studied in monographs and survey articles [3, 4]. Early refinements focused on sharpening the two-sided estimate by inserting additional terms. A well-known Bullen-type refinement, established by Tseng et al. and further developed by Bouharket et al. [21], states that for a convex function f ,

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \\ &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \\ &\leq \frac{f(a)+f(b)}{2} \quad (3.2) \end{aligned}$$

This refinement shows that additional evaluations of f at symmetric interior points can tighten the classical bounds. Dragomir and Agarwal [2] complemented this line by deriving error estimates for the midpoint and trapezoidal approximations in terms of the first derivative. For example, if $|f'|$ is convex, then

$$\begin{aligned} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|) \quad (3.3) \end{aligned}$$

These early results established the template for later modifications: either strengthen the convexity assumption, change the integral operator, or incorporate derivative information.

3.2 Generalized convexity approaches

A major branch of prior work replaces ordinary convexity by weaker or parameterized convexity notions. For s -convex functions in the second sense [6, 7, 23], the inequality becomes

$$\begin{aligned} 2^{s-1} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{s+1}, s \\ &\in (0, 1] \quad (3.4) \end{aligned}$$

When $s=1$, (4) reduces to (1). For h -convex functions [8, 9], the upper bound takes the form

$$\frac{1}{b-a} \int_a^b f(x) dx \leq (f(a) + f(b)) \int_0^1 h(t) dt \quad (3.5)$$

where $h: [0,1] \rightarrow \mathbb{R}$ is a nonnegative auxiliary function. Choosing $h(t) = t$ recovers the classical case, while $h(t) = t^s$ yields the s -convex version.

Other convexity classes have also been treated. For harmonically convex functions [10], the midpoint is replaced by the harmonic mean:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2} \quad (3.6)$$

For quasi-convex functions [11, 37], the upper bound becomes

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \max\{f(a), f(b)\} \quad (3.7)$$

which is weaker than the convex bound but applies to a larger class of functions. For m -convex and (α, m) -convex functions, Özdemiř et al. [30] and Bakula et al. [13] obtained estimates for the right-hand side of (1) using second derivatives. Log-convex and r -convex functions were studied by Gill et al. [29], leading to bounds involving the logarithmic mean $L(f(a), f(b))$:

$$\frac{1}{b-a} \int_a^b f(x) dx \leq L(f(a), f(b)) \quad (3.8)$$

Coordinated log-convex functions on rectangles were considered by Alomari and Darus [28], who developed Hadamard-type and Jensen-type inequalities in two dimensions. More recently, P -functions and semi P -functions have been introduced to capture functions satisfying $f(tx + (1-t)y) \leq f(x) + f(y)$ or a scaled variant [35]. For semi P -functions, the Hermite–Hadamard-type inequality becomes

$$\begin{aligned} \frac{1}{a+b} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq (a+b) \frac{f(a) + f(b)}{2} \quad (3.9) \end{aligned}$$

These generalized convexity approaches share a common goal: extend the applicability of (1) beyond ordinary convex functions while preserving a two-sided integral estimate.

3.3 Fractional integral approaches

The introduction of fractional calculus provided a natural framework for non-local generalizations of (1). Using Riemann–Liouville fractional integrals J_{a+}^α and J_{b-}^α of order $\alpha > 0$, Sarikaya et al. [16] established the fractional Hermite–Hadamard inequality

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \\ &\leq \frac{f(a) + f(b)}{2} \quad (3.10) \end{aligned}$$

For $\alpha = 1$, (10) reduces to (1). Sarikaya et al. [16] also derived an integral identity for differentiable mappings and obtained fractional error bounds analogous to (3). Anastassiou et al. [17] developed Montgomery identities for fractional integrals and used them to prove Ostrowski- and Grüss-type inequalities. Sarikaya and Ogunmez [34] extended these identities and obtained further Ostrowski-type estimates for differentiable convex functions. Dahmani [33] established Chebyshev-type fractional integral inequalities for synchronous functions, while Dahmani et al. studied reverse Minkowski and Grüss fractional inequalities. Iscan and Kunt [36] combined fractional integrals with quasi-geometrically convex functions and Fejér-type weights. These fractional approaches are particularly relevant for systems with memory, non-local interactions, and anomalous diffusion [14, 15].

3.4 Derivative-based and error-bound approaches

A parallel line of research focuses on estimating the gap between the integral average and the midpoint or trapezoidal bounds using derivative information. Dragomir and Agarwal [2] introduced the identity

$$\begin{aligned} \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \\ = \frac{b-a}{2} \int_0^1 (1-2t) f'(ta) \\ + (1-t)b) dt \quad (3.11) \end{aligned}$$

which serves as the starting point for many error bounds. Pearce and Pečarić [22] extended (3) to L^q spaces, while Kirmaci et al. [23] treated ss -convex derivatives. Özdemiř et al. [30] derived estimates for functions whose second derivatives in absolute value are mm -convex. Sarikaya and Ogunmez [34] obtained Ostrowski-type inequalities via fractional integrals under convexity assumptions. Iscan et al. [37] studied nn -times differentiable quasi-convex functions and established several new integral inequalities using Hölder and power-mean inequalities. Iscan et al. [38] extended these ideas to preinvex and prequasiinvex functions on invex sets. These derivative-based approaches are especially useful in numerical quadrature, where explicit error control is required.

3.5 Weighted and Fejér-type approaches

Weighted generalizations of (1) were initiated by Fejér [24]. If $g: [a, b] \rightarrow [0, \infty)$ is integrable and symmetric about the midpoint, i.e. $g(a+b-x) = g(x)$, then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx &\leq \frac{1}{b-a} \int_a^b f(x) g(x) dx \\ &\leq \frac{f(a) + f(b)}{2} \int_a^b g(x) dx \quad (3.12) \end{aligned}$$

Sarikaya [25] and Set et al. [27] developed Fejér-type inequalities via fractional integrals. Bakula and Pečarić [18] obtained weighted generalizations involving nonlinear transformations of the base interval and product functions. Iscan and Kunt [36] established Hermite–Hadamard–Fejér inequalities for quasi-geometrically convex functions using fractional integrals. These weighted forms are important when different points of the interval carry different importance, as in probability, statistics, and signal processing.

3.6 Interval subdivision and composite refinements

Another practical approach is to subdivide the interval and apply (1) on each subinterval. Bouharket et al. [21] developed a systematic refinement by dividing $[a, b]$ into 2^n equal subintervals symmetric about the midpoint. They defined sequences

$$L_n = \frac{1}{2^n} \sum_{k=1}^{2^n} f\left(\frac{(2^{n+1} - 2k + 1)a + (2k - 1)b}{2^{n+1}}\right)$$

$$R_n = \frac{1}{2^n} \left[\sum_{k=1}^{2^n-1} f\left(\frac{(2^n - k)a + kb}{2^n}\right) + \frac{f(a) + f(b)}{2} \right]$$

and proved the chain

$$L_0 = f\left(\frac{a+b}{2}\right) \leq L_1 \leq \dots \leq L_n \leq \frac{1}{b-a} \int_a^b f(x) dx \leq R_n$$

$$\leq \dots \leq R_1 \leq R_0 = \frac{f(a) + f(b)}{2}$$

This refinement yields composite midpoint and trapezoidal bounds that converge to the integral average as $n \rightarrow \infty$. It provides a direct bridge between the Hermite–Hadamard inequality and composite numerical quadrature rules [2, 20]. Related composite rules were studied by Dragomir and Wang [19] and Cerone and Dragomir [20] in the context of trapezoidal and Ostrowski-type error estimates.

3.7 Product-function and Chebyshev-type approaches

Several authors have extended (1) to products of two functions. Pachpatte (as cited in [18, 23, 31]) established Hadamard-type inequalities for products of convex functions. Bakula and Pečarić [18] obtained weighted generalizations involving two convex functions and nonlinear transformations. Kirmaci et al. [23] proved product inequalities for convex and s -convex functions. Set et al. [31] developed further inequalities for products of two functions using Minkowski, Hölder, and Young inequalities. Set et al. [27] also considered products of log-convex, m -convex, and $(\alpha, m)(\alpha, m)$ -convex functions. Dahmani [33] established Chebyshev-type fractional integral inequalities for synchronous functions. These product-function approaches are relevant to covariance bounds, Grüss-type inequalities, and applications in probability and statistics.

3.8 Non-local and variable-order extensions

Beyond fractional integrals, variable-order differential operators have been introduced to model systems whose order of differentiation changes with the

independent or dependent variable [39]. Although these developments are not always directly framed as Hermite–Hadamard inequalities, they represent a natural next step for non-local and history-dependent extensions. The variable-order operator proposed by Coimbra and studied by Diaz and Coimbra [39] has been applied to viscoelastic oscillators and van der Pol-type equations. Combining such operators with Hermite–Hadamard-type inequalities remains a largely open research direction.

3.9 Summary and identified gaps

The prior approaches reviewed above have substantially extended the scope of the classical Hermite–Hadamard inequality. Generalized convexity has broadened the class of admissible functions; fractional integrals have introduced non-local weighting; derivative-based estimates have provided explicit error control; weighted and Fejér-type forms have allowed non-uniform importance; interval subdivision has produced convergent composite bounds; and product-function inequalities have addressed multiplicative structures.

Nevertheless, several gaps remain. First, the literature is highly fragmented: different modifications are often studied in isolation, and unified comparisons are rare. Second, the computational implications of these modifications—particularly in numerical integration, error analysis, and algorithm selection—are frequently underemphasized. Third, there is no systematic guidance on choosing an appropriate modified inequality for a given function class, available derivative information, or desired accuracy. Fourth, fractional and variable-order versions require further numerical validation and benchmarking against classical quadrature rules.

The present paper addresses these gaps by providing a unified analytical review of modified Hermite–Hadamard inequalities, comparing their sharpness and computational cost through benchmark examples, and offering practical selection criteria for analysts and numerical practitioners.

IV. OUR APPROACH

The present study adopts a unified analytical and computational approach to the study of modified Hermite–Hadamard inequalities. Unlike prior works that typically focus on a single generalization or refinement, we systematically integrate multiple strands of research—generalized convexity, fractional integrals, derivative-based error bounds, weighted Fejér forms, and interval subdivision—into a single comparative framework. This enables both a theoretical synthesis and a practical assessment of their relative strengths and limitations. The approach is designed to answer three central questions: (i) how do different modifications relate to the classical inequality and to each other; (ii) how sharp are their bounds in concrete numerical settings; and (iii) which modification is most appropriate for a given function class and computational task.

4.1 Unified analytical framework

We begin by establishing a common notation and by reducing each modification to a canonical form. The classical Hermite–Hadamard inequality for a convex function $f: [a, b] \rightarrow \mathbb{R}$ is written as

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2} \quad (4.1)$$

Each modification is then interpreted as arising from a change in one of three components: the convexity condition, the integral operator, or the weighting function. This reduction facilitates direct comparison. For example:

- **Generalized convexity:** replacing ordinary convexity by s -convexity, h -convexity, harmonic convexity, quasi-convexity, or m -convexity yields modified bounds such as

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{s+1} \quad (4.2)$$

$$\frac{1}{b-a} \int_a^b f(x) dx \leq (f(a)+f(b)) \int_0^1 h(t) dt \quad (4.3)$$

And

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a)+f(b)}{2} \quad (4.4)$$

Fractional integrals: replacing the Riemann integral by Riemann–Liouville fractional integrals J_{a+}^α and J_{b-}^α gives

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \leq \frac{f(a)+f(b)}{2} \quad (4.5)$$

Weighted forms: introducing a symmetric weight $g(a+b-x) = g(x)$ yields the Fejér-type inequality

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \leq \frac{1}{b-a} \int_a^b f(x) g(x) dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x) dx \quad (4.5)$$

Interval subdivision: partitioning $[a, b]$ into 2^n symmetric subintervals and applying (1) on each subinterval produces the refinement

$$L_n \leq \frac{1}{b-a} \int_a^b f(x) dx \leq R_n$$

where L_n and R_n are the composite midpoint and trapezoidal bounds. This yields monotone sequences that converge to the integral average.

By expressing all modifications in this common framework, we are able to compare their assumptions, their bounds, and their reduction to the classical case in a systematic manner.

4.2 Computational benchmarking methodology

To evaluate the practical performance of each modification, we employ a computational benchmarking

methodology. Four representative test functions are selected:

polynomial: $f_1(x) = x$ on $[0, 2]$;

exponential: $f_2(x) = e^x$ on $[0, 1]$;

reciprocal: $f_3(x) = \frac{1}{x}$ on $[1, 4]$;

radical: $f_4(x) = \sqrt{x}$ on $[0, 4]$.

For each function, we compute the lower bound L , the integral average I , and the upper bound U for each modification. The following error metrics are used:

$$E_L = I - L, \quad E_U = U - I, \quad W = U - L, \quad R = \frac{I - L}{U - L}, \quad 0 \leq R \leq 1 \quad (2.13)$$

Here W measures the width of the bounding interval; smaller W indicates a sharper inequality. The ratio R describes the relative position of the true integral average between the lower and upper bounds. For composite refinements, we compute the bounds for $n = 1, 2, 4, 8$ subintervals and observe convergence. For fractional versions, we vary α over $0.5, 1, 1.5, 2, 0.5, 1, 1.5, 2$ and record the effect on the bounds. For Fejér-type inequalities, we test symmetric weights such as $g(x) = 1$, $g(x) = x(a+b-x)$, and $g(x) = \sin(\pi(x-a)/(b-a))$. All numerical computations are performed with high precision, and relative errors are reported to avoid scale dependence.

Numerical Computations and Results

In this section, we present detailed numerical computations for the four benchmark functions introduced in Section 3.2. For each function, we compute the lower bound L , the integral average I , the upper bound U , the error metrics E_L , E_U , W , and R , and where applicable, derivative-based bounds and composite refinements. These results illustrate the sharpness and practical behavior of the modified Hermite–Hadamard inequalities.

Example 1: $f_1(x) = x^2$ on $[0, 2]$

The function $f_1(x) = x^2$ is strictly convex since $f''(x) = 2 > 0$. We compute:

$$L = f\left(\frac{0+2}{2}\right) = f(1) = 1$$

$$I = \frac{1}{2-0} \int_0^2 x^2 dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2 = \frac{4}{3} \approx 1.3333333$$

$$U = \frac{f(0)+f(2)}{2} = \frac{0+4}{2} = 2$$

Thus, the classical inequality holds:

$$1 \leq \frac{4}{3} \leq 2$$

Error metrics:

$$E_L = I - L = \frac{4}{3} - 1 = \frac{1}{3} = 0.3333333$$

$$E_U = U - I = 2 - \frac{4}{3} = \frac{2}{3} = 0.6666667$$

$$W = U - L = 2 - 1 = 1$$

$$R = \frac{I - L}{U - L} = \frac{\frac{1}{3}}{1} = \frac{1}{3} \approx 0.3333333$$

Derivative-based bounds: Since $f''(x) = 2$, the maximum $M=2$. The trapezoidal and midpoint error bounds are:

$$\frac{M(b-a)^2}{12} = \frac{2 \cdot 2^2}{12} = \frac{2}{3} \approx 0.666667$$

$$\frac{M(b-a)^2}{24} = \frac{2 \cdot 2^2}{24} = \frac{1}{3} \approx 0.333333$$

these bounds coincide exactly with the actual errors.

Composite bounds: Dividing the interval into n equal subintervals, the composite midpoint M_n and trapezoidal T_n bounds are given in the table below for $n=1,2,4,8$

n	M_n	T_n	I	$E_L^{(n)} = I - M_n$	$E_U^{(n)} = T_n - I$	$W_n = T_n - M_n$
1	1.000000	2.000000	1.333333	0.333333	0.666667	1.000000
2	1.250000	1.500000	1.333333	0.083333	0.166667	0.250000
4	1.312500	1.375000	1.333333	0.020833	0.041667	0.062500
8	1.328125	1.343750	1.333333	0.005208	0.010417	0.015625

Example 2: $f_1(x) = e^x$ on $[0,1]$

The function $f_1(x) = e^x$ is strictly convex since $f''(x) = e^x > 0$. We compute:

$$L = f\left(\frac{0+1}{2}\right) = e^{0.5} \approx 1.648721$$

$$I = \frac{1}{1-0} \int_0^1 e^x dx = e - 1 \approx 1.718282$$

$$U = \frac{f(0) + f(1)}{2} = \frac{1 + e}{2} = 1.859141$$

Thus, the classical inequality holds:

$$1.648721 \leq 1.718282 \leq 1.859141$$

Error metrics:

$$E_L = 0.069561$$

$$E_U = 0.140859$$

$$W = 0.210420$$

$$R = \frac{0.069561}{0.210420} \approx 0.3306$$

Derivative-based bounds: The maximum

$M = e \approx 2.718282$. The bounds are

$$\frac{e(1)^2}{12} \approx 0.226524, \frac{e(1)^2}{24} \approx 0.113262.$$

The actual errors are smaller than these bounds.

Composite bounds:

n	M_n	T_n	I	$E_L^{(n)}$	$E_U^{(n)}$	W_n
1	1.648721	1.859141	1.718282	0.069561	0.140859	0.210420
2	1.700513	1.753931	1.718282	0.017769	0.035649	0.053418
4	1.713815	1.727222	1.718282	0.004467	0.008940	0.013407
8	1.717161	1.720519	1.718282	0.001121	0.002237	0.003358

Example 3: $f_1(x) = \frac{1}{x}$ on $[1,4]$

The function $f_1(x) = \frac{1}{x}$ is strictly convex since $f''(x) = \frac{2}{x^3} > 0$. We compute:

$$L = f\left(\frac{1+4}{2}\right) = f(2.5) = 0.4$$

$$I = \frac{1}{4-1} \int_1^4 \frac{1}{x} dx = \frac{\ln 4}{3} \approx 0.462098$$

$$U = \frac{f(1) + f(4)}{2} = \frac{1 + 0.25}{2} = 0.625$$

Thus, the classical inequality holds:

$$0.4 \leq 0.462098 \leq 0.625$$

Error metrics:

$$E_L = 0.062098$$

$$E_U = 0.162902$$

$$W = 0.225$$

$$R \approx 0.276$$

Derivative-based bounds: $f''(x) = \frac{2}{x^3}$ the maximum on $[1,4]$ occurs at $x=1$, so $M=2$. The bounds are:

$$\frac{2(3)^2}{12} = 1.5, \frac{2(3)^2}{24} = 0.75.$$

which are very conservative.

Composite bounds:

n	M_n	T_n	I	$E_L^{(n)}$	$E_U^{(n)}$	W_n
1	0.400000	0.625000	0.462098	0.062098	0.162902	0.225000
2	0.439560	0.512500	0.462098	0.022538	0.050402	0.072940
4	0.455387	0.476030	0.462098	0.006711	0.013932	0.020643
8	0.460312	0.465709	0.462098	0.001786	0.003611	0.005397

Example 2: $f_1(x) = \sqrt{x}$ on $[0,4]$

- The function $\sqrt{x}$ is concave in the classical sense, so the classical Hermite–Hadamard inequality does not hold (it reverses). However, it is s -convex in the second sense for $s = 1/2$. The s -convex bounds are:

$$L_s = 2^{s-1} f\left(\frac{0+4}{2}\right) = 2^{-\frac{1}{2}} \sqrt{2} = 1$$

$$I = \frac{1}{4} \int_0^4 \sqrt{x} dx \approx 1.333333$$

$$U = \frac{f(0) + f(4)}{s+1} = \frac{0 + 2}{1 + 0.5} = \frac{4}{3} \approx 1.333333$$

Thus, the classical inequality holds:

$$1 \leq \frac{4}{3} \leq \frac{4}{3}$$

Error metrics:

$$E_L = I - L_s = \frac{4}{3} - 1 = \frac{1}{3} \approx 0.333333$$

$$E_U = U_s - I = \frac{4}{3} - \frac{4}{3} = 0$$

$$W = U_s - L_s = \frac{1}{3} \approx 0.333333$$

Note: In this example, the upper bound coincides exactly with the integral average, and the s -convex inequality provides a sharp bound.

Summary Table of Results

The following table provides a general comparison of the bounds and error metrics for the four benchmark functions.

Function	Interval	L	I	U	E_L	E_U	W	R
x^2	$[0, 2]$	1.000000	1.333333	2.000000	0.333333	0.666667	1.000000	0.3333
e^x	$[0, 1]$	1.648721	1.718282	1.859141	0.069561	0.140859	0.210420	0.3306
$1/x$	$[1, 4]$	0.400000	0.462098	0.625000	0.062098	0.162902	0.225000	0.2760
$\sqrt{x}$ ($s = 1/2$)	$[0, 4]$	1.000000	1.333333	1.333333	0.333333	0.000000	0.333333	1.0000

Concluding Remarks on Numerical Results

The numerical results demonstrate that:

- For ordinary convex functions, the classical inequality holds, and interval subdivision rapidly brings the bounds closer to the integral average.
- Derivative-based bounds for functions with bounded second derivatives control the actual error well, although they may be conservative.
- For $f(x) = 1/x$, the classical inequality leads to the well-known relation between harmonic, logarithmic, and arithmetic means.
- For $f(x) = x$, which is concave in the classical sense, the s -convex version with $s = 1/2$ provides a sharp bound and demonstrates the advantage of generalizations.
- In general, no single modification is universally best; the choice of the appropriate form depends on the function class, available derivative information, and required accuracy.

4.3 Comparative selection criteria

Based on the analytical properties and numerical results, we develop a comparative selection framework. The following criteria are used:

1. Applicability: the class of functions for which the inequality is valid.
2. Sharpness: the width W of the bounding interval and the closeness of L and U to I .
3. Computational cost: the amount of derivative, fractional, or quadrature information required.
4. Parameter sensitivity: the dependence of the bounds on parameters such as ss , h , α , or the weight g .

The comparison yields practical guidelines. For ordinary convex functions, the classical inequality is often sufficient. For functions satisfying weaker convexity conditions, ss -convex, h -convex, or quasi-convex bounds are appropriate. For reciprocal and rate-type quantities, harmonically convex versions are natural. For non-local or memory-dependent models, fractional versions are preferred. For non-uniform importance, Fejér-type inequalities are used. For high-accuracy numerical integration, interval subdivision and derivative-based bounds provide the most direct improvements.

4.4 Novel contributions

The novel contributions of this study are fourfold:

1. **Systematic integration:** We bring together generalized convexity, fractional integrals, derivative-based bounds, weighted forms, and interval subdivision into a single comparative framework. This allows, for the first time, a

direct assessment of their relative sharpness and computational cost under uniform conditions.

2. **Computational evaluation:** We provide a quantitative benchmarking of the most widely used modifications using representative test functions. This goes beyond the usual theoretical presentation and offers practical insight into the actual accuracy of each bound.
3. **Selection framework:** We propose a decision framework that helps analysts and numerical practitioners choose the most appropriate modification for a given function class, available derivative information, and desired accuracy.
4. **Identification of open problems:** We highlight gaps in the literature, including the need for unified error estimates, the comparison of fractional and classical bounds in numerical quadrature, and the extension of variable-order operators to Hermite–Hadamard-type inequalities.

By combining analytical synthesis with computational evaluation, this approach provides a comprehensive resource for researchers, graduate students, and practitioners interested in the theory and applications of Hermite–Hadamard-type inequalities.

V. CONCLUSION

The Hermite–Hadamard inequality stands as one of the most elegant and widely applicable results in convex analysis, providing a fundamental link between the value of a convex function at the midpoint of an interval, its integral average, and the arithmetic mean of its endpoint values. Since its discovery in the late nineteenth century, it has inspired a vast body of research aimed at refining, generalizing, and extending its applicability to broader classes of functions, more general integral operators, and increasingly complex analytical settings. This paper has provided a unified analytical review of the most significant modifications of the Hermite–Hadamard inequality and has evaluated their computational performance through a systematic benchmarking methodology.

The comparative analysis yielded practical selection guidelines. For ordinary convex functions with sufficient regularity, the classical Hermite–Hadamard inequality or its derivative-based refinements are recommended. When the function exhibits generalized convexity, the corresponding modified inequality should be chosen: ss -convex bounds for ss -convex functions, h -convex bounds for h -convex functions, harmonically convex bounds for functions involving reciprocals, and quasi-convex bounds for functions with weaker convexity properties. For non-local or memory-dependent phenomena, fractional versions are appropriate. When different points of the interval have different importance, Fejér-type inequalities with symmetric weights should be used. For high-precision numerical integration, interval

subdivision combined with derivative-based error control provides the most effective strategy.

The computational results also highlighted that no single modification is universally superior. The choice of inequality involves a trade-off between sharpness, computational cost, and the strength of the required assumptions. For example, quasi-convex bounds are widely applicable but often less sharp than convex bounds, while fractional bounds are highly flexible but require the evaluation of fractional integrals and may introduce additional numerical complexity. These trade-offs must be carefully considered in practical applications.

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