

Establishment of Ostrowski-Type Inequalities for Multiplicatively Trigonometric Convex Functions

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ABSTRACT

In this study, we present new and refined forms of Ostrowski-type inequalities for multiplicatively trigonometric convex functions. The main objective of this research is to derive generalized versions of the classical Ostrowski inequality that are applicable to this class of functions. This theoretical and analytical study is motivated by the proofs of inequalities for convex functions. In this work, the properties of multiplicatively trigonometric convexity, together with analytical methods, are employed to obtain novel and generalized results. Specifically, under the assumption that the absolute value of the derivative is bounded and satisfies the multiplicatively trigonometric convexity condition, new Ostrowski-type bounds are established. Moreover, by employing Hölder's inequality together with the properties of multiplicatively trigonometric convexity, further refined and generalized forms of these inequalities are derived. In addition, the resulting integral expressions are shown to be expressible in terms of the modified Bessel function of order zero, which enhances the analytical interpretation of the main results and facilitates their use in practical computations. In the final part of this study, we demonstrate the applicability of our results by showing that Ostrowski-type inequalities can also be used to establish new bounds and relationships among special means of real numbers, including arithmetic, geometric, harmonic, logarithmic, P-logarithmic, and identric means. The obtained results may find potential applications in applied mathematics and numerical analysis.

Keywords- Bessel function; Hölder's inequality; multiplicatively trigonometric convex functions; Ostrowski inequality; trigonometric convex functions.

I. INTRODUCTION

Integral inequalities constitute one of the fundamental areas of mathematical analysis, investigating the relationships between the pointwise values of functions, their integral means, and error bounds. Such inequalities play a crucial role in numerical analysis and approximation methods, particularly in evaluating errors arising from numerical integration processes. One of the classical and fundamental results in this field is the Ostrowski inequality, first introduced in 1937 and later generalized in various directions by researchers such as Dragomir [1], making it applicable to different classes of functions.

One of the most important factors contributing to the advancement of inequality theory in mathematical analysis is the concept of convexity, which has been extensively studied by Niculescu and Persson [2]. With its analytical and geometric generalizations, various forms of convexity have emerged in recent years. For instance, the concept of multiplicative convexity introduced by Ali et al.[3] provides a powerful framework for nonlinear analysis and systems involving multiplicative dynamics. On the other hand, Kadakal [4] introduced Hermite–Hadamard-type inequalities for trigonometric convex functions, which are applicable in modeling phenomena involving periodic behavior.

Although significant progress has been achieved independently in these areas, the concept of multiplicative

trigonometric convexity (MTC) was introduced and defined in 2025. However, the development of Ostrowski-type integral inequalities and bounds for this class of functions has not yet been fully explored, representing an important research gap in this field. Therefore, by utilizing the newly defined MTC functions, this study presents, for the first time, new and generalized versions of Ostrowski-type integral inequalities with rigorous proofs. In this work, Hölder's inequality and the modified Bessel function of order zero are employed. Finally, the obtained results are applied to establish new inequalities among arithmetic, geometric, harmonic, and logarithmic special means, demonstrating their theoretical and practical significance in applied mathematics and numerical analysis.

II. METHODOLOGY AND PRELIMINARIES

This study is entirely based on theoretical and analytical mathematical approaches, focusing on the establishment of integral inequalities for multiplicatively trigonometric convex functions. All results are obtained through rigorous mathematical reasoning, algebraic computations, and techniques from the theory of integral inequalities.

2.1 Research Design

The present work is a theoretical study aimed at establishing Ostrowski-type integral inequalities for multiplicatively trigonometric convex functions and applying the obtained results to different means of real numbers.

2.2 Materials

The following mathematical concepts are employed in this paper:

- Fundamental concepts of real analysis
- Multiplicatively trigonometric convex functions
- Integral inequalities
- Classical Ostrowski inequality
- Modified Bessel function of order zero

Lemma 1. Let $\varphi: J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J^0 , where $a, b \in J$ with $a < b$. If $\varphi' \in L[a, b]$, then the following equality holds [5].

$$\begin{aligned} \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \\ = \frac{(z-a)^2}{b-a} \int_0^1 t \varphi'(tz + (1-t)a) dt \\ - \frac{(b-z)^2}{b-a} \int_0^1 t \varphi'(tz + (1-t)b) dt \end{aligned}$$

Definition 1. Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable function. We say that the derivative of f is bounded on $[a, b]$ if there exists a constant $M > 0$ such that $|f'(x)| \leq M$ for every $x \in [a, b]$ [1].

Definition 2. Let $I \subseteq \mathbb{R}$ be an interval and let $F: I \rightarrow (0, \infty)$ be a positive function. Then F is called a multiplicatively trigonometric convex (MTC) function if the following inequality holds for every $x, y \in I$ and every $t \in [0, 1]$ [6].

$$\begin{aligned} F(tx + (1-t)y) &\leq (F(x))^{\sin(\frac{\pi t}{2})} (F(y))^{\cos(\frac{\pi t}{2})} \\ \text{or} \\ \ln f(tx + (1-t)y) &\leq \sin \frac{\pi t}{2} \ln f(x) + \cos \frac{\pi t}{2} \ln f(y) \end{aligned}$$

Theorem 1. Let $p > 1$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If φ and θ are real-valued functions defined on the interval $[u, v]$, and $|\varphi|^p$ and $|\theta|^q$ are integrable functions on $[u, v]$, then the following inequality is known as Hölder's inequality [7].

$$\begin{aligned} \int_u^v |\varphi(t)\theta(t)| dt \\ \leq \left(\int_u^v |\varphi(t)|^p dt \right)^{\frac{1}{p}} \left(\int_u^v |\theta(t)|^q dt \right)^{\frac{1}{q}} \end{aligned}$$

Theorem 2 (Ostrowski inequality). Let $g: [a, b] \rightarrow \mathbb{R}$ be a differentiable function on $[a, b]$ with a bounded derivative, that is $|g'(x)| \leq M, x \in [a, b]$. Then, for every $x \in [a, b]$, the following inequality holds [8].

$$\begin{aligned} |g(x) - \frac{1}{b-a} \int_a^b g(t) dt| \\ \leq \frac{(x-a)^2 + (b-x)^2}{2(b-a)} M \end{aligned}$$

Definition 3. The special function defined by the following integral representation is called the modified Bessel function of order zero [9].

$$I_0(x) = \frac{1}{\pi} \int_0^\pi e^{x \cdot \sin \theta} d\theta$$

2.3 Methodological Framework

First, new and refined forms of Ostrowski-type integral inequalities for multiplicatively trigonometric convex functions are established, and their proofs are developed based on direct arguments and integral techniques. Subsequently, these results are applied to different means of real numbers. Moreover, the newly obtained and improved forms are compared with previously established results.

2.4 Data Sources

This study does not employ tables, experimental data, graphs, or statistical databases. All results were obtained through theoretical analysis and rigorous mathematical proofs.

III. FINDINGS

In this section, we present new and refined forms of Ostrowski-type integral inequalities for multiplicatively trigonometric convex functions. In the

final part, these concepts are applied to special means of real numbers, which may have broad applications in the field of numerical analysis.

Theorem 3. Let $\varphi: J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J^0 , where $a, b \in J$ with $a < b$. Suppose that $|\varphi'|$ is a multiplicatively trigonometric convex function on $[a, b]$, $\varphi' \in L[a, b]$, and $|\varphi'| \leq k$ for every $z \in [a, b]$, where $k \geq 1$. Then the following inequality holds:

$$\left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right| \leq \frac{1}{2} e^{\sqrt{2}(\ln k)} \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right]$$

Proof:

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right| \\ &= \left| \frac{(z-a)^2}{b-a} \int_0^1 t \varphi'(tz + (1-t)a) dt - \frac{(b-z)^2}{b-a} \int_0^1 t \varphi'(tz + (1-t)b) dt \right| \\ &\leq \frac{(z-a)^2}{b-a} \int_0^1 t |\varphi'(tz + (1-t)a)| dt + \frac{(b-z)^2}{b-a} \int_0^1 t |\varphi'(tz + (1-t)b)| dt \\ &\leq \frac{(z-a)^2}{b-a} \int_0^1 t |\varphi'(z)|^{\sin \frac{\pi t}{2}} |\varphi'(a)|^{\cos \frac{\pi t}{2}} dt + \frac{(b-z)^2}{b-a} \int_0^1 t |\varphi'(z)|^{\sin \frac{\pi t}{2}} |\varphi'(b)|^{\cos \frac{\pi t}{2}} dt \\ &\leq \frac{(z-a)^2}{b-a} \int_0^1 t \cdot k^{\sin \frac{\pi t}{2}} \cdot k^{\cos \frac{\pi t}{2}} dt + \frac{(b-z)^2}{b-a} \int_0^1 t \cdot k^{\sin \frac{\pi t}{2}} \cdot k^{\cos \frac{\pi t}{2}} dt \\ &= \frac{(z-a)^2}{b-a} \int_0^1 t \cdot k^{\sin \frac{\pi t}{2} + \cos \frac{\pi t}{2}} dt + \frac{(b-z)^2}{b-a} \int_0^1 t \cdot k^{\sin \frac{\pi t}{2} + \cos \frac{\pi t}{2}} dt \\ &\text{if } \frac{\pi t}{2} = x \Rightarrow t = \frac{2}{\pi} x \Rightarrow dt = \frac{2}{\pi} dx, t \rightarrow 0 \Rightarrow x \rightarrow 0, t \rightarrow 1 \Rightarrow x \rightarrow \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} &= \underbrace{\frac{(z-a)^2}{b-a} \int_0^{\frac{\pi}{2}} \frac{2}{\pi} t \cdot k^{\sin t + \cos t} \cdot \frac{2}{\pi} dt}_{I_1} \\ &+ \underbrace{\frac{(b-z)^2}{b-a} \int_0^{\frac{\pi}{2}} \frac{2}{\pi} t \cdot k^{\sin t + \cos t} \cdot \frac{2}{\pi} dt}_{I_2} \dots \dots \dots (1) \end{aligned}$$

Now, we introduce a series of important transformations:

$$\begin{aligned} \sin t + \cos t &= \sqrt{2} \sin \left(t + \frac{\pi}{4} \right) \\ a \sin t + b \cos t &= \sqrt{a^2 + b^2} \sin(t + \alpha) \\ \Rightarrow k^{\sin t + \cos t} &= k^{\sqrt{2} \sin \left(t + \frac{\pi}{4} \right)} \\ &= e^{\sqrt{2}(\ln k) \sin \left(t + \frac{\pi}{4} \right)} \dots \dots \dots * \end{aligned}$$

Considering relation (*), we have:

$$\begin{aligned} I_1 &= \int_0^{\frac{\pi}{2}} \frac{4}{\pi^2} t \cdot k^{\sin t + \cos t} dt \\ &= \frac{4}{\pi^2} \int_0^{\frac{\pi}{2}} t \cdot e^{\sqrt{2}(\ln k) \cdot \sin \left(t + \frac{\pi}{4} \right)} dt \dots \dots \dots ** \end{aligned}$$

Considering the following transformations, we have:

$$\begin{aligned} \sin \left(t + \frac{\pi}{4} \right) &\leq 1 \\ \sqrt{2}(\ln k) \cdot \sin \left(t + \frac{\pi}{4} \right) &\leq \sqrt{2}(\ln k) \\ e^{\sqrt{2}(\ln k) \cdot \sin \left(t + \frac{\pi}{4} \right)} &\leq e^{\sqrt{2}(\ln k)}, k \geq 1 \dots \dots \dots *** \end{aligned}$$

Substituting (***) into (**), we have:

$$\begin{aligned} I_1 &= \frac{4}{\pi^2} \int_0^{\frac{\pi}{2}} t \cdot e^{\sqrt{2}(\ln k) \cdot \sin \left(t + \frac{\pi}{4} \right)} dt \\ &\leq \frac{4}{\pi^2} \int_0^{\frac{\pi}{2}} t \cdot e^{\sqrt{2}(\ln k)} dt \\ &= e^{\sqrt{2}(\ln k)} \cdot \frac{4}{\pi^2} \cdot \frac{1}{2} t^2 \Big|_0^{\frac{\pi}{2}} = e^{\sqrt{2}(\ln k)} \cdot \frac{4}{\pi^2} \cdot \frac{1}{2} \cdot \frac{\pi^2}{4} \\ &= \frac{1}{2} e^{\sqrt{2}(\ln k)} \end{aligned}$$

$$\Rightarrow I_1 \leq \frac{1}{2} e^{\sqrt{2}(\ln k)}$$

Similarly, I_2 can be calculated as follows:

$$\begin{aligned} I_2 &= \int_0^{\frac{\pi}{2}} \frac{4}{\pi^2} t \cdot k^{\sin t + \cos t} dt \\ &= \frac{4}{\pi^2} \int_0^{\frac{\pi}{2}} t \cdot e^{\sqrt{2}(\ln k) \cdot \sin \left(t + \frac{\pi}{4} \right)} dt \dots \dots \dots **** \end{aligned}$$

Substituting (***) into (****), we have:

$$I_2 = \frac{4}{\pi^2} \int_0^{\frac{\pi}{2}} t \cdot e^{\sqrt{2}(\ln k) \cdot \sin\left(t + \frac{\pi}{4}\right)} dt \leq \frac{4}{\pi^2} \int_0^{\frac{\pi}{2}} t \cdot e^{\sqrt{2}(\ln k)} dt$$

$$= e^{\sqrt{2}(\ln k)} \cdot \frac{4}{\pi^2} \cdot \frac{1}{2} t^2 \Big|_0^{\frac{\pi}{2}} = e^{\sqrt{2}(\ln k)} \cdot \frac{4}{\pi^2} \cdot \frac{1}{2} \cdot \frac{\pi^2}{4}$$

$$= \frac{1}{2} e^{\sqrt{2}(\ln k)}$$

$$\Rightarrow I_2 \leq \frac{1}{2} e^{\sqrt{2}(\ln k)}$$

Substituting the values of I_1 and I_2 into equation (1), we obtain:

$$\left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right|$$

$$\leq \frac{(z-a)^2}{b-a} \cdot \frac{1}{2} e^{\sqrt{2}(\ln k)} + \frac{(b-z)^2}{b-a} \cdot \frac{1}{2} e^{\sqrt{2}(\ln k)}$$

$$= \frac{1}{2} e^{\sqrt{2}(\ln k)} \left[\frac{(z-a)^2}{b-a} + \frac{(b-z)^2}{b-a} \right]$$

$$= \frac{1}{2} e^{\sqrt{2}(\ln k)} \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right]$$

$$\Rightarrow \left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right|$$

$$\leq \frac{1}{2} e^{\sqrt{2}(\ln k)} \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right]$$

Theorem 4. Let $\varphi: J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J^0 , where $a, b \in J$ with $a < b$. Suppose that $|\varphi'|^2$ is a multiplicatively trigonometric convex function on $[a, b]$, and $|\varphi'(t)| < k$ for every $\forall t \in [a, b]$, where $k > 0$. Moreover, let $p, q > 1$ satisfy the condition $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequality holds:

$$\left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right|$$

$$\leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q\sqrt{2}\sin\left(t + \frac{\pi}{4}\right) \cdot \ln k} dt \right)^{\frac{1}{q}}$$

$$\cdot \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right]$$

Proof:

$$\left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right|$$

$$= \left| \frac{(z-a)^2}{b-a} \int_0^1 t \cdot \varphi'(tz + (1-t)a) dt - \frac{(b-z)^2}{b-a} \int_0^1 t \cdot \varphi'(tz + (1-t)b) dt \right|$$

$$\leq \underbrace{\frac{(z-a)^2}{b-a} \int_0^1 t \cdot \varphi'(tz + (1-t)a) dt}_{I_1} + \underbrace{\frac{(b-z)^2}{b-a} \int_0^1 t \cdot \varphi'(tz + (1-t)b) dt}_{I_2}$$

The term I_1 is considered as follows:

$$I_1 = \int_0^1 |t \cdot \varphi'(tz + (1-t)a)| dt$$

$$\leq \left(\int_0^1 |t|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |\varphi'(tz + (1-t)a)|^q dt \right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\int_0^1 |\varphi'(tz + (1-t)a)|^q dt \right)^{\frac{1}{q}}$$

$$\leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\int_0^1 (|\varphi'(z)|^q)^{\sin \frac{\pi t}{2}} (|\varphi'(a)|^q)^{\cos \frac{\pi t}{2}} dt \right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\int_0^1 |\varphi'(z)|^q \sin^{\frac{\pi t}{2}} (|\varphi'(a)|^q)^{\cos \frac{\pi t}{2}} dt \right)^{\frac{1}{q}}$$

$$\leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\int_0^1 k^q \cdot \sin^{\frac{\pi t}{2}} \cdot k^q \cdot \cos^{\frac{\pi t}{2}} dt \right)^{\frac{1}{q}}$$

if $x = \frac{\pi t}{2} \Rightarrow dx = \frac{\pi}{2} dt \Rightarrow \frac{2}{\pi} dx = dt$

if $t \rightarrow 0 \Rightarrow x \rightarrow 0, t \rightarrow 1 \Rightarrow x \rightarrow \frac{\pi}{2}$

$$= \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} k^q \cdot \sin^q + \cos^q dt \right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} k^q (\sin + \cos) dt \right)^{\frac{1}{q}}$$

$\sin + \cos = \sqrt{2} \sin\left(t + \frac{\pi}{4}\right)$

$$\begin{aligned} &\Rightarrow \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} k^{q\sqrt{2}\sin\left(t+\frac{\pi}{4}\right)} dt\right)^{\frac{1}{q}} \\ &= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q\sqrt{2}\sin\left(t+\frac{\pi}{4}\right) \cdot \ln k} dt\right)^{\frac{1}{q}} \\ &\Rightarrow I_1 \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q\sqrt{2}\sin\left(t+\frac{\pi}{4}\right) \cdot \ln k} dt\right)^{\frac{1}{q}} \end{aligned}$$

Similarly, for (I_2) , we have:

$$\begin{aligned} I_2 &= \int_0^1 |t \cdot \varphi'(tz + (1-t)b)| dt \\ &\leq \left(\int_0^1 |t|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 |\varphi'(tz + (1-t)b)|^q dt\right)^{\frac{1}{q}} \\ &= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\int_0^1 |\varphi'(tz + (1-t)b)|^q dt\right)^{\frac{1}{q}} \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\int_0^1 (|\varphi'(z)|^q)^{\sin\frac{\pi t}{2}} (|\varphi'(b)|^q)^{\cos\frac{\pi t}{2}} dt\right)^{\frac{1}{q}} \\ &= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\int_0^1 |\varphi'(z)|^q \sin^{\frac{\pi t}{2}} (|\varphi'(b)|^q)^{\cos\frac{\pi t}{2}} dt\right)^{\frac{1}{q}} \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\int_0^1 k^{q \cdot \sin\frac{\pi t}{2}} \cdot k^{q \cdot \cos\frac{\pi t}{2}} dt\right)^{\frac{1}{q}} \\ \text{if } x &= \frac{\pi t}{2} \Rightarrow dx = \frac{\pi}{2} dt \Rightarrow \frac{2}{\pi} dx = dt \\ \text{if } t &\rightarrow 0 \Rightarrow x \rightarrow 0, t \rightarrow 1 \Rightarrow x \rightarrow \frac{\pi}{2} \\ &= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} k^{q \cdot \sin t + q \cdot \cos t} dt\right)^{\frac{1}{q}} \\ &= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} k^{q(\sin t + \cos t)} dt\right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned} \sin t + \cos t &= \sqrt{2} \sin\left(t + \frac{\pi}{4}\right) \\ &\Rightarrow \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} k^{q\sqrt{2}\sin\left(t+\frac{\pi}{4}\right)} dt\right)^{\frac{1}{q}} \\ &= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q\sqrt{2}\sin\left(t+\frac{\pi}{4}\right) \cdot \ln k} dt\right)^{\frac{1}{q}} \\ &\Rightarrow I_2 \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q\sqrt{2}\sin\left(t+\frac{\pi}{4}\right) \cdot \ln k} dt\right)^{\frac{1}{q}} \end{aligned}$$

Substituting the values of I_1 and I_2 into the above expression, we obtain:

$$\begin{aligned} &\left|\varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt\right| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q\sqrt{2}\sin\left(t+\frac{\pi}{4}\right) \cdot \ln k} dt\right)^{\frac{1}{q}} \\ &\quad \cdot \left[\frac{(z-a)^2 + (b-z)^2}{b-a}\right] \end{aligned}$$

Remark1. The integral $\int_0^{\frac{\pi}{2}} e^{q\sqrt{2} \cdot \ln k \cdot \sin\left(t+\frac{\pi}{4}\right)} dt$ can be evaluated numerically or expressed in terms of the modified Bessel function of order zero as follows:

Remark 2. If the above integral is expressed in terms of the modified Bessel function of order zero, then the following inequality holds:

$$\begin{aligned} &\left|\varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt\right| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(2 \cdot I_0(q\sqrt{2} \ln k)\right)^{\frac{1}{q}} \left[\frac{(z-a)^2 + (b-z)^2}{b-a}\right] \end{aligned}$$

Or

$$\begin{aligned} &\left|\varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt\right| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(2 \cdot I_0(\ln k^{q\sqrt{2}})\right)^{\frac{1}{q}} \left[\frac{(z-a)^2 + (b-z)^2}{b-a}\right] \end{aligned}$$

Proof:

By considering the integral $\int_0^{\frac{\pi}{2}} e^{q\sqrt{2} \cdot \ln k \cdot \sin\left(t+\frac{\pi}{4}\right)} dt$ and its representation in terms of the modified Bessel function of order zero, we obtain:

Modifid Beseel Function Of Order Zero: $I_0(x) =$

$$\frac{1}{\pi} \int_0^\pi e^{x \cdot \sin \theta} d\theta$$

$$\int_0^{\frac{\pi}{2}} e^{q \sqrt{2} \cdot \ln k \cdot \sin(t + \frac{\pi}{4})} dt$$

Setting $x = q\sqrt{2}\ln k$ we obtain:

$$\int_0^{\frac{\pi}{2}} e^{x \cdot \sin(t + \frac{\pi}{4})} dt$$

if $\theta = t + \frac{\pi}{4} \Rightarrow d\theta = dt$

$$t \rightarrow 0 \Rightarrow \theta \rightarrow \frac{\pi}{4}, t \rightarrow \frac{\pi}{2} \Rightarrow \theta \rightarrow \frac{3\pi}{4}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} e^{x \cdot \sin(t + \frac{\pi}{4})} dt = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{x \cdot \sin \theta} d\theta$$

We know that $[\frac{\pi}{4}, \frac{3\pi}{4}] \subset [0, \pi]$ Moreover, if $f(x)$ is a positive function, then its integral increases when the interval of integration is enlarged. Since $e^{x \cdot \sin \theta}$ is positive for all θ , it follows that:

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{x \cdot \sin \theta} d\theta < \int_0^\pi e^{x \cdot \sin \theta} d\theta$$

$$\Rightarrow \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{q \sqrt{2} \cdot \ln k \cdot \sin \theta} d\theta$$

$$< \int_0^\pi e^{q \sqrt{2} \cdot \ln k \cdot \sin \theta} d\theta$$

Moreover, we know that:

$$I_0(x) = \frac{1}{\pi} \int_0^\pi e^{x \cdot \sin \theta} d\theta$$

$$I_0(q \cdot \sqrt{2} \ln k) = \frac{1}{\pi} \int_0^\pi e^{q \sqrt{2} \ln k \cdot \sin \theta} d\theta$$

$$\Rightarrow \pi \cdot I_0(q \cdot \sqrt{2} \ln k) = \int_0^\pi e^{q \sqrt{2} \ln k \cdot \sin \theta} d\theta$$

$$\Rightarrow \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{q \sqrt{2} \cdot \ln k \cdot \sin \theta} d\theta < \pi \cdot I_0(q \cdot \sqrt{2} \ln k)$$

$$\int_0^{\frac{\pi}{2}} e^{q \sqrt{2} \cdot \ln k \cdot \sin(t + \frac{\pi}{4})} dt = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{q \sqrt{2} \cdot \ln k \cdot \sin \theta} d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} e^{q \sqrt{2} \cdot \ln k \cdot \sin(t + \frac{\pi}{4})} dt < \pi \cdot I_0(q \cdot \sqrt{2} \ln k)$$

Hence, we can write

$$\left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right|$$

$$\leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q \sqrt{2} \cdot \ln k \cdot \sin(t + \frac{\pi}{4})} dt \right)^{\frac{1}{q}}$$

$$\cdot \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right]$$

$$\leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{\pi} \cdot \pi \cdot I_0(q \cdot \sqrt{2} \ln k) \right)^{\frac{1}{q}}$$

$$\cdot \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right]$$

$$\Rightarrow \left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right|$$

$$\leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} (2$$

$$\cdot I_0(q \cdot \sqrt{2} \ln k))^{\frac{1}{q}}$$

$$\cdot \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right]$$

3.1 Applications to Special Means

In this section, we use the following notations for the special means of two nonnegative numbers a and b with $b > a$.

1. Arithmetic Mean: The arithmetic mean a and b is defined as follows:

$$A = A(a, b) = \frac{a+b}{2}, a, b \geq 0$$

2. Geometric Mean: The geometric mean of a and b is defined as follows:

$$G = G(a, b) = \sqrt{ab}, a, b \geq 0$$

3. Harmonic Mean: The harmonic mean of a and b is given as follows:

$$H = H(a, b) = \frac{2ab}{a+b}, a, b \geq 0$$

4. Logarithmic Mean: The logarithmic mean of a and b is defined as follows:

$$L = L(a, b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, a \neq b \\ a, a = b \end{cases} a, b > 0$$

5. P-Logarithmic Mean: The pp p-logarithmic mean of a and b is defined as follows:

$$L_p = L_p(a, b)$$

$$= \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}, a \neq b, p \in \mathbb{R} - \{-1, 0\} \\ a, a = b, b > 0 \end{cases}$$

6. Identric Mean: The identric mean a and b is defined as follows:

$$I = I(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}}, a, b > 0$$

These special means have wide applications in numerical approximation and in numerous other areas of

mathematics. Moreover, the following simple relations among these means have been previously established.

$$H \leq G \leq L \leq I \leq A$$

It is also known that the L_p mean is a monotonically increasing function with respect to the real parameter p . In particular, $L_0 = I, L_{-1} = L$. Proposition 1. Assume that $0 < a < b$. Then, for $k > 1$, the following inequality holds:

$$|A^n(a, b) - L_n^n(a, b)| \leq e^{\ln k \sqrt{2}} \frac{b-a}{4}$$

Proof: By setting $z = \frac{a+b}{2}$ in Theorem 3, and using the function $\varphi(x) = x^n$ for $x \in (0, \infty)$, we have:

$$\text{Theorem 3: } \left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right| \leq \frac{1}{2} e^{\sqrt{2} \ln k} \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right]$$

$$\varphi(z) = \varphi\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2}\right)^n = (A(a, b))^n = A^n(a, b)$$

$$\begin{aligned} \frac{1}{b-a} \int_a^b \varphi(t) dt &= \frac{1}{b-a} \int_a^b t^n dt = \frac{1}{b-a} \frac{t^{n+1}}{n+1} \Big|_a^b \\ &= \frac{1}{(b-a)(n+1)} (b^{n+1} - a^{n+1}) \\ &= \frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)} \\ &= (L_n(a, b))^n = L_n^n(a, b) \\ \frac{(z-a)^2 + (b-z)^2}{b-a} &= \frac{\left(\frac{b+a}{2} - a\right)^2 + \left(b - \frac{a+b}{2}\right)^2}{b-a} \\ &= \frac{\left(\frac{a+b-2a}{2}\right)^2 + \left(\frac{2b-a-b}{2}\right)^2}{b-a} = \frac{\left(\frac{b-a}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2}{b-a} \\ &= \frac{2\left(\frac{b-a}{2}\right)^2}{b-a} = \frac{b-a}{2} \end{aligned}$$

Substituting these values into the inequality of the theorem, we obtain:

$$\Rightarrow |A^n(a, b) - L_n^n(a, b)| \leq e^{\ln k \sqrt{2}} \cdot \frac{b-a}{4}$$

Proposition 2. Assume that $0 < a < b$. Then, for $k > 1$, the following inequality holds:

$$|A^{-1}(a, b) - L^{-1}(a, b)| \leq e^{\ln k \sqrt{2}} \cdot \frac{b-a}{4}$$

Proof: By setting $z = \frac{a+b}{2}$ in Theorem 3, and using the function $\varphi(x) = x^{-1}$ for $x \in (0, \infty)$, we have:

$$\text{Theorem 3: } \left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right| \leq \frac{1}{2} e^{\sqrt{2} \ln k} \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right]$$

$$\varphi(z) = \varphi\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2}\right)^{-1} = [A(a, b)]^{-1} = A^{-1}(a, b)$$

$$\begin{aligned} \frac{1}{b-a} \int_a^b \varphi(t) dt &= \frac{1}{b-a} \int_a^b \frac{1}{t} dt = \frac{1}{b-a} \ln t \Big|_a^b \\ &= \frac{\ln b - \ln a}{b-a} \\ &= \left(\frac{b-a}{\ln b - \ln a}\right)^{-1} = [L(a, b)]^{-1} = L^{-1}(a, b) \\ \frac{(z-a)^2 + (b-z)^2}{b-a} &= \frac{\left(\frac{a+b}{2} - a\right)^2 + \left(b - \frac{a+b}{2}\right)^2}{b-a} \\ &= \frac{\left(\frac{a+b-2a}{2}\right)^2 + \left(\frac{2b-a-b}{2}\right)^2}{b-a} = \frac{\left(\frac{b-a}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2}{b-a} \\ &= \frac{2\left(\frac{b-a}{2}\right)^2}{b-a} = \frac{b-a}{2} \end{aligned}$$

Substituting these values into the inequality of the theorem, we obtain:

$$\Rightarrow |A^{-1}(a, b) - L^{-1}(a, b)| \leq e^{\ln k \sqrt{2}} \cdot \frac{b-a}{4}$$

Proposition 3. Let $0 < a < b$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$.

1. Then, for $k > 0$, the following inequality holds:

$$\begin{aligned} &|A^n(a, b) - L_n^n(a, b)| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q \sqrt{2} \ln k \sin(t + \frac{\pi}{4})} dt\right)^{\frac{1}{q}} \cdot \frac{b-a}{2} \end{aligned}$$

Proof: By applying Theorem 4 and using the function $f(x) = x^n$, we obtain:

$$\begin{aligned} \text{Theorem 4: } &\left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right| \\ &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q \sqrt{2} \ln k \sin(t + \frac{\pi}{4})} dt\right)^{\frac{1}{q}} \\ &\cdot \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right] \end{aligned}$$

As shown in Proposition 1, for the function $\varphi(x) = x^n$,

$$\varphi(z) = \varphi\left(\frac{a+b}{2}\right) = A^n(a, b)$$

$$\begin{aligned} \frac{1}{b-a} \int_a^b \varphi(t) dt &= \frac{1}{b-a} \int_a^b t^n dt = L_n^n(a, b) \\ \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right] &= \frac{b-a}{2} \\ \Rightarrow |A^n(a, b) - L_n^n(a, b)| &\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q \sqrt{2} \ln k \sin(t + \frac{\pi}{4})} dt\right)^{\frac{1}{q}} \cdot \frac{b-a}{2} \end{aligned}$$

Proposition 4. Let $0 < a < b$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$.

1. Then, for $k > 0$, the following inequality holds:

$$|A^{-1}(a, b) - L^{-1}(a, b)|$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q\sqrt{2} \ln k \sin(t+\frac{\pi}{4})} dt\right)^{\frac{1}{q}} \cdot \frac{b-a}{2}$$

Proof:

Theorem4: $\left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right|$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q\sqrt{2} \ln k \sin(t+\frac{\pi}{4})} dt\right)^{\frac{1}{q}} \cdot \left[\frac{(z-a)^2(b-z)^2}{b-a} \right]$$

Furthermore, by using the function $\varphi(x) = \frac{1}{x}$ for $x \in (0, \infty)$ and setting $z = \frac{a+b}{2}$, we obtain:

$$\varphi(x) = x^{-1}$$

$$\varphi(z) = \varphi\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2}\right)^{-1} = A^{-1}(a, b)$$

$$\frac{1}{b-a} \int_a^b \varphi(t) dt = \frac{1}{b-a} \int_a^b \frac{1}{t} dt = L^{-1}(a, b)$$

$$\frac{(z-a)^2 + (b-z)^2}{b-a} = \frac{b-a}{2}$$

$$\Rightarrow |A^{-1}(a, b) - L^{-1}(a, b)|$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{q\sqrt{2} \ln k \sin(t+\frac{\pi}{4})} dt\right)^{\frac{1}{q}} \cdot \frac{b-a}{2}$$

Proposition 5. Let $0 < a < b$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $k > 0$, the following inequality holds:

$$|A^n(a, b) - L_n^n(a, b)|$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(2 \cdot I_0(q \cdot \sqrt{2} \ln k)\right)^{\frac{1}{q}} \cdot \frac{b-a}{2}$$

Proof: By setting $z = \frac{a+b}{2}$ in Remark2, and using the function $\varphi(x) = x^n$ for $x \in (0, \infty)$, we obtain:

Remark2: $\left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(2 \cdot I_0(q \cdot \sqrt{2} \ln k)\right)^{\frac{1}{q}} \cdot \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right]$

For $\varphi(x) = x^n$,

$$\varphi(z) = \varphi\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2}\right)^n = (A(a, b))^n$$

$$= A^n(a, b)$$

$$\frac{1}{b-a} \int_a^b \varphi(t) dt = \frac{1}{b-a} \int_a^b t^n dt = \frac{1}{b-a} \cdot \frac{t^{n+1}}{n+1} \Big|_a^b$$

$$= \frac{1}{(b-a)(n+1)} (b^{n+1} - a^{n+1}) = \frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}$$

$$= (L_n(a, b))^n = L_n^n(a, b)$$

$$\frac{(z-a)^2 + (b-z)^2}{b-a} = \frac{\left(\frac{b+a}{2} - a\right)^2 + \left(b - \frac{a+b}{2}\right)^2}{b-a}$$

$$= \frac{\left(\frac{a+b-2a}{2}\right)^2 + \left(\frac{2b-a-b}{2}\right)^2}{b-a} = \frac{\left(\frac{b-a}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2}{b-a}$$

$$= \frac{\frac{2(b-a)^2}{4}}{b-a} = \frac{b-a}{2}$$

Substituting these values into the inequality given in Remark 2, we obtain:

$$\Rightarrow |A^n(a, b) - L_n^n(a, b)|$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(2 \cdot I_0(q \cdot \sqrt{2} \ln k)\right)^{\frac{1}{q}} \cdot \frac{b-a}{2}$$

Proposition 6. Let $0 < a < b$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $k > 0$, the following inequality holds:

$$|A^{-1}(a, b) - L^{-1}(a, b)|$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(2 \cdot I_0(q \cdot \sqrt{2} \ln k)\right)^{\frac{1}{q}} \cdot \frac{b-a}{2}$$

Proof: By setting $z = \frac{a+b}{2}$ in Remark2, and using the function $\varphi(x) = x^{-1}$ for $x \in (0, \infty)$, we obtain:

Remark: $\left| \varphi(z) - \frac{1}{b-a} \int_a^b \varphi(t) dt \right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(2 \cdot I_0(q \cdot \sqrt{2} \ln k)\right)^{\frac{1}{q}} \cdot \left[\frac{(z-a)^2 + (b-z)^2}{b-a} \right]$

For $\varphi(x) = x^{-1}$,

$$\varphi(z) = \varphi\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2}\right)^{-1} = [A(a, b)]^{-1}$$

$$= A^{-1}(a, b)$$

$$\frac{1}{b-a} \int_a^b \varphi(t) dt$$

$$= \frac{1}{b-a} \int_a^b \frac{1}{t} dt = \frac{1}{b-a} \ln t \Big|_a^b$$

$$= \frac{\ln b - \ln a}{b-a}$$

$$= \left(\frac{b-a}{\ln b - \ln a}\right)^{-1} = [L(a, b)]^{-1} = L^{-1}(a, b)$$

$$\frac{(z-a)^2 + (b-z)^2}{b-a} = \frac{\left(\frac{a+b}{2} - a\right)^2 + \left(b - \frac{a+b}{2}\right)^2}{b-a}$$

$$= \frac{\left(\frac{a+b-2a}{2}\right)^2 + \left(\frac{2b-a-b}{2}\right)^2}{b-a} = \frac{\left(\frac{b-a}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2}{b-a}$$

$$= \frac{\frac{2(b-a)^2}{4}}{b-a} = \frac{b-a}{2}$$

Substituting these values into the inequality given in Remark 2, we obtain:

$$\Rightarrow |A^{-1}(a, b) - L^{-1}(a, b)|$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(2 \cdot I_0(q \cdot \sqrt{2} \ln k)\right)^{\frac{1}{q}} \cdot \frac{b-a}{2}$$

IV. DISCUSSION

The results of this study demonstrate that new and refined forms of Ostrowski-type inequalities can be established based on multiplicatively trigonometric convexity. In Theorem 3, a new inequality was derived under the condition that the absolute value of the derivative is bounded, which represents a new and generalized form of the classical Ostrowski inequality. The scientific significance of this result lies in the fact that classical Ostrowski inequalities generally consider only the boundedness of the absolute value of the derivative, whereas in the present study the property of multiplicatively trigonometric convexity is additionally incorporated [6].

In Theorem 4, by employing Hölder's inequality and the properties of multiplicatively trigonometric convexity, further improved and generalized forms of Ostrowski-type inequalities have been obtained, extending the scope of application of the results. This result has a more general structure compared with Theorem 3 because the presence of the parameters p and q allows the inequality to be adapted to various analytical conditions. This feature increases its significance in the fields of applied mathematics and numerical analysis [7]. Another important achievement of this study is that, in Remark 2, the integral is expressed in a simplified form using the modified Bessel function of order zero, which enhances the analytical clarity of the main result and facilitates its use in practical computations. This also highlights the value of employing special functions in the analysis of integral inequalities [9].

The application of the obtained results to special means demonstrates that these inequalities possess not only theoretical significance but also practical importance for analyzing the relationships among arithmetic, geometric, harmonic, logarithmic, P -logarithmic, and identric means, with potential applications in numerical analysis.

The results of Propositions 1, 2, 3, 4, 5, and 6 further clarify the applied aspects of this topic. These findings show that new bounds for special means can be obtained through Ostrowski-type inequalities, and the relationships among various means can be expressed more precisely. This subject holds particular importance in the theory of inequalities, since special means have wide applications in numerous branches of mathematical analysis, numerical methods, and applied sciences.

The classical Ostrowski inequality provides a bound for the distance between the pointwise value of a function and its integral mean; however, it does not take into account the specific structural properties of convexity. Subsequently, numerous researchers have presented various forms of Ostrowski-type inequalities for convex functions [10], S -convex functions [11], multiplicative convex functions [12], harmonically convex functions [13], P -convex functions [14], and trigonometric convex functions [15]. Furthermore, the

concept of multiplicative trigonometric convexity has recently been employed in the context of Hermite–Hadamard-type inequalities; however, its application to Ostrowski-type inequalities has remained limited to date. To fill this gap, new bounds have been established in this study. Consequently, these results make a valuable contribution to addressing the existing scientific gap and open a new analytical avenue for the advancement of the theory of integral inequalities.

Compared with existing studies, the main distinctive feature of the present research is that new forms of Ostrowski-type inequalities are obtained by employing the concept of multiplicatively trigonometric convexity, and these results are subsequently applied to special means of real numbers. In particular, by taking $x = \frac{a+b}{2}$ and using the functions $\varphi(x) = x^n$ and $\varphi(x) = x^{-1}$ in Theorem 3, Propositions 1 and 2 are respectively obtained. Similarly, by applying Theorem 4 with the same functions and setting $x = \frac{a+b}{2}$, Propositions 3 and 4 are derived. Furthermore, by using Remark 2 with $\varphi(x) = x^n$ and $\varphi(x) = x^{-1}$ under the same substitution, Propositions 5 and 6 are obtained, respectively. These results have potential applications in the field of numerical analysis.

Overall, this study represents a significant advancement in the theory of Ostrowski-type inequalities and establishes new Ostrowski-type inequalities for multiplicatively trigonometric convex functions. Through their application to special means of real numbers, these results open important avenues in the field of numerical analysis. Future research could extend Ostrowski-type inequalities to other generalized convexity structures and integral inequalities as well.

V. CONCLUSION

In this study, new and refined forms of Ostrowski-type inequalities have been obtained for multiplicatively trigonometric convex functions. By employing the properties of multiplicatively trigonometric convexity and analytical tools, particularly Hölder's inequality, new bounds have been established to accurately estimate the difference between the pointwise value of a function and its integral mean. The obtained results demonstrate that the concept of multiplicatively trigonometric convexity provides an effective analytical framework for the development of Ostrowski-type inequalities.

The scientific value of this research lies in the fact that new and refined forms of Ostrowski-type inequalities have been established based on the concept of multiplicatively trigonometric convexity, thereby offering new possibilities for the further development of the theory of integral inequalities.

Furthermore, the results obtained based on Hölder's inequality and the representation of the integral through the modified Bessel function of order zero

demonstrate that this framework has not only theoretical significance but also practical value for computational applications and numerical analysis. The applications to special means, including arithmetic, geometric, harmonic, logarithmic, p -logarithmic, and identric means, indicate that the obtained inequalities have potential applicability in various areas of numerical analysis.

This study makes a valuable contribution to the existing research on convexity and integral inequalities, demonstrating that the concept of multiplicatively trigonometric convexity can also be employed to extend other classical inequalities. Therefore, it is anticipated that these results will serve as a useful reference for future research in numerical analysis, integral approximation, and applied mathematics.

In future studies, similar methods may be applied to other generalized convexity structures, fractional integrals, fractional derivatives, and Jensen and Simpson-type inequalities. Moreover, further analytical and numerical aspects of multiplicatively trigonometric convex functions can be considered as important and promising research directions for future investigations.

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